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# Symmetries and deformations in the spherical shell model

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## Abstract

We discuss symmetries of the spherical shell model that make contact with the geometric collective model of Bohr and Mottelson. The most celebrated symmetry of this kind is  $SU(3)$ , which is the basis of Elliott's model of rotation. It corresponds to a deformed mean field induced by a quadrupole interaction in a single major oscillator shell  $N$  and can be generalized to include several major shells. As such, Elliott's  $SU(3)$  model establishes the link between the spherical shell model and the (quadrupole component of the) geometric collective model. We introduce the analogue symmetry induced by an octupole interaction in two major oscillator shells  $N - 1$  and  $N$ , leading to an octupole-deformed solution of the spherical shell model. We show that in the limit of large oscillator shells,  $N \rightarrow \infty$ , the algebraic octupole interaction tends to that of the geometric collective model.

## 1. Introduction

Our understanding of the structure of the atomic nucleus is at present incomplete, based as it is on various models with limited ranges of applicability. Examples are the spherical shell model and the geometric collective model. While the former stresses the single-particle nature of the nucleons in the nucleus, their coherent motion is emphasized by the latter. A recurring question in more than half a century of basic nuclear research has been how to reconcile two such opposing views of the structure of the nucleus. The currently accepted paradigm is that the structural patterns and regularities, as predicted by the geometric collective model, arise as emergent behaviour from the complex many-body problem of nucleonic interactions in the context of the spherical shell model.

While this connection can, perhaps, be established in principle, and much current work is going on along these lines, many of its aspects still remain unclear. In this paper we study this question from the perspective of symmetries. Ever since the pioneering studies by Wigner, Racah and Elliott, symmetry considerations have played a pivotal role in the development of nuclear models, in particular of the spherical shell model. We do not intend to review here all such symmetries but focus on those that make contact with the geometric collective model.

An alternative microscopic approach to nuclear collective motion that is under intense current investigation involves the use of self-consistent mean-field models [1], whereby nuclei are described variationally from realistic

interactions between their constituent nucleons. There has also been much recent work to take these models beyond mean field [2]. Models based on symmetries provide exactly solvable limits for the same collective features described by these mean-field approaches, but at the cost of using semi-realistic, rather than fully realistic, hamiltonians. Nevertheless, because of the exact solvability of these symmetry-based models, unique perspectives on collective properties can often be identified.

We start by painting a qualitative picture of the two models we aim connect in this paper in sections (1.1) and (1.2). One symmetry of the spherical shell model is  $SU(3)$ , which forms the basis of our understanding of nuclear rotation. It can be given a rigorous formulation in the context of Wigner's supermultiplet model, which is reviewed in section 2. Elliott's  $SU(3)$  model of quadrupole deformation is discussed in section 3, first in its elementary version applied to a single major shell of the harmonic oscillator and subsequently in its extended version applied to several shells. Section 4 deals with the analogous problem of octupole deformation and proposes a corresponding symmetry in the context of the spherical shell model. Some concluding remarks are made in section 5.

### 1.1. A shell-model primer

In first approximation the structure of a nucleus is determined by the nuclear mean field, that is, the average potential felt by each nucleon resulting from the interactions with all others. In particular, the observed shell structure of nuclei can be understood on the basis of the notion of mean field. However, for a detailed description of many nuclear properties, a residual nucleon-nucleon interaction on top of the average potential must be taken into account. Nuclear structure at low energies is particularly affected by the residual interaction between nucleons in the valence shell, that is, in the outermost shell that is not completely filled.

While a current focus of nuclear structure theory aims at a consistent microscopic derivation of the nuclear mean field and the residual interaction, several of the basic features of the structure of nuclei are captured in the follow-

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ing schematic nuclear hamiltonian:

$$\hat{H} = \sum_{k=1}^A \left( \frac{p_k^2}{2m_n} + \frac{1}{2} m_n \omega^2 r_k^2 + \zeta_{\ell\ell} \ell_k^2 + \zeta_{\ell s} \bar{\ell}_k \cdot \bar{s}_k \right) + \sum_{1 \leq k < l}^A \hat{V}_{ri}(\xi_k, \xi_l), \quad (1)$$

where  $A$  is the atomic mass number (the number of nucleons in the nucleus),  $\xi_k$  is a short-hand notation for the spatial coordinates, the spin and isospin variables of nucleon  $k$ ,  $\xi_k \equiv \{\bar{r}_k, \bar{s}_k, \bar{t}_k\}$ , and  $m_n$  is the nucleon mass. The first term in equation (1) is the kinetic energy of the nucleons. The second term is a harmonic-oscillator potential with frequency  $\omega$ , which is governed by the nuclear size and which is a crude approximation to the nuclear mean field [3] for well-bound nuclei. A more realistic nuclear mean field, (*e.g.*, a Woods–Saxon potential) does not display the degeneracy of states with different orbital angular momentum  $\ell$  in the same major shell, characteristic of a harmonic oscillator. To some extent this deficiency of the harmonic-oscillator potential can be remedied by adding an  $\ell^2$  orbit–orbit term, which lifts the  $\ell$ -degeneracy and gives rise to a single-particle spectrum more in accord with that from experiment. The fourth term in the hamiltonian (1) corresponds to a spin–orbit coupling in the nucleonic motion, whose assumption was the decisive step in the justification of the nuclear shell model [4, 5] by providing a natural explanation of the observed ‘magic’ numbers, those neutron  $N$  or proton  $Z$  numbers for which the nucleus acquires an increased stability. The fact that the spin–orbit interaction is strong gives rise to another very important feature in nuclear structure, namely that the higher major shells contain orbitals that intrude from the next oscillator shell and have the opposite parity from the others in the major shell. The last term in equation (1) is the residual two-body interaction. It depends in a complex fashion on the mean field and on the valence space that is made available to the nucleons. Because of its dependence on the space in which it acts, it is often referred to as an *effective* interaction.

If single-particle energy spacings are large compared to a typical matrix element of the residual interaction, nucleons move independently. This limit corresponds to the shell model of independent particles. The neglect of the last term in the hamiltonian (1) leads to uncorrelated many-particle eigenstates that are Slater determinants constructed from the single-particle eigenfunctions of the harmonic oscillator. Slater determinants involve products of single-particle states, organized so that the wave function is fully anti-symmetric under particle interchange, as required by the Pauli Principle. This is the key to the shell structure exhibited by nuclei. If the residual interaction is not neglected, a true many-body problem results, where it is critical to incorporate configuration mixing of the Slater determinants resulting from the residual interaction. Usually it is only necessary to include configuration mixing within a single major shell for neutrons and a single major shell for protons, except fairly near shell closure, where coherent excitations from other major shells can be lowered into the region of low-lying states of the dominant shell and must therefore be considered on

the same footing.

A good approximation to the residual effective interaction for use in a shell-model treatment of nuclei involves a pairing interaction  $\hat{V}_{\text{pairing}}$ , acting between pairs of alike nucleons in time-reversed orbits, and a sum of separable two-body interactions acting between all nucleons, *viz.*

$$\hat{V}_{ri} = \hat{V}_{\text{pairing}} + \sum_J \alpha_J \hat{P}_J^\dagger \cdot \hat{P}_J, \quad (2)$$

where  $\hat{P}_J^\dagger$  creates a coherent particle–hole pair with multipolarity  $J$  and parity  $(-)^J$ . In fairly light nuclei the connection between modern realistic effective interactions and a schematic sum over separable interactions has been carefully demonstrated [6].

The term with  $J = 0$ , called the monopole interaction, has the primary effect of evolving the single-particle energies of the nuclear mean field, sometimes even changing the order of single-particle levels and the magic numbers [6]. The other terms govern the mixing of simple shell-model configurations that give rise to the wide variety of features seen in nuclei across the periodic table.

The pairing component of the residual interaction,  $\hat{V}_{\text{pairing}}$  in equation (2), has far-reaching consequences. Perhaps most importantly, it gives rise to pairing correlations, which have a pervasive impact on nuclear structure properties throughout the periodic table. For the purposes of this discussion, however, the pairing interaction, as well as the related delta interaction,  $\delta(\bar{r}_1 - \bar{r}_2)$ , conserve total orbital angular momentum  $L$  and total spin  $S$ , besides total angular momentum  $J$  associated with rotational invariance. When this term dominates, it leads to a classification called *LS* (or Russell–Saunders) coupling. This is strongly broken, however, by the spin–orbit term in the nuclear mean field, which favours *jj* coupling. The conflict between *LS* coupling and *jj* coupling plays a crucial role in determining the structure of the nucleus, as was recognized and studied in the earliest days of the nuclear shell model [7]. The generally accepted conclusion is that *jj* coupling is relevant for the vast majority of nuclei while the *LS* classification is appropriate for the very lightest nuclei only [8].

Under certain circumstances a residual three-body interaction must also be taken into account [9]. In very light nuclei, the effects of three-body forces are especially important and they must be incorporated fully [10]. In heavier nuclei their effects are less pronounced and can typically be absorbed into the mean field, analogous to the two-body monopole interaction [11].

A second important feature that plays a critical role in dictating the structure of the nucleus is the number of active neutrons and protons in the valence shell(s). As noted earlier, the residual interaction between identical nucleons has a pairing character which favours the formation of pairs of nucleons in time-reversed orbits. This is no longer true if the valence space contains both neutrons and protons, in which case there is a strong attraction in *all* orbits resulting primarily through the quadrupole interaction [ $J = 2$  in equation (2)] but with contributions under appropriate circumstances from other multipoles. Hence, nuclei display a wide range of possible spectra that can vary from pairing-type to rotational-like. The evolution from one type to the other is governed by the product of

the number of neutrons with the number of protons in the valence shell [12].

In heavier nuclei, *i.e.* in the rare-earth and actinide regions, the mixing between orbitals with different parities becomes increasingly more important. In these regimes important octupole-like ( $J = 3$ ) correlations arise, and it is typically necessary to incorporate them along with pairing and quadrupole correlations.

## 1.2. A collective-model primer

In 1879, in a study of the properties of a droplet of incompressible liquid, Lord Rayleigh showed [13] that its normal modes of vibration may be described by the variables  $\alpha_{\lambda\mu}$  which appear in the expansion of the droplet's radius,

$$R(\theta, \phi) = R_0 \left( 1 + \sum_{\lambda\mu} \alpha_{\lambda\mu} Y_{\lambda\mu}^*(\theta, \phi) \right), \quad (3)$$

where  $Y_{\lambda\mu}(\theta, \phi)$  are spherical harmonics in terms of the spherical angles  $\theta$  and  $\phi$ . In spite of some obvious differences between a quantized atomic nucleus and a classical liquid drop, the latter has been used to describe the properties of the atomic nucleus since the pioneering work of von Weizsäcker [14] and Bohr and Kalckar [15]. Since then it has been customary for nuclear physicists to adopt the multipole parameterization (3) to describe vibrations of the nuclear fluid. There were several key steps and observations that followed soon after these original works. Feenberg [16] and Bohr and Wheeler [17] studied the shape and stability of a deformed and of a charged liquid drop. A few years later Flügge [18] noted that nuclear rotations may produce rotational spectra, though he did not yet appreciate that surface vibrations and rotations may be related to one another. These ideas eventually culminated in the geometric collective model, introduced in the classical papers by Rainwater [19], Bohr [20], and Bohr and Mottelson [21].

The geometric collective model complements the spherical shell model by emphasizing the coherent behaviour of many nucleons, including quadrupole and higher-multipole deformations as well as rotations and vibrations that involve a large portion of the nucleus [22, 23].

A key contribution of Bohr [20] and Bohr and Mottelson [21] was to note that the static shape and orientation of a deformed nucleus and the collective deformation variables of a spherical nucleus were related to each other. Flügge had already suggested [18] that these collective variables could play the role of dynamical variables but it was Bohr [20] who extended this from a classical to a quantum drop. In this picture the  $\alpha_{\lambda\mu}$  in the nuclear surface (3) are considered as (time-dependent) variables that determine the shape of the nuclear surface. For particular choices of  $\lambda$  different shapes result. This is illustrated in figure 1, where the quadrupole case ( $\lambda = 2$ ) is shown as well as examples of octupole ( $\lambda = 3$ ) and hexadecapole ( $\lambda = 4$ ) deformation.

For quadrupole deformations ( $\lambda = 2$ ), which dominate in most regions of the periodic table, the hamiltonian can be written as

$$\hat{H} = \hat{T} + \hat{V} = \frac{1}{2B} \sum_{\mu} (\pi_{2\mu})^2 + \frac{1}{2} C \sum_{\mu} (\alpha_{2\mu})^2, \quad (4)$$

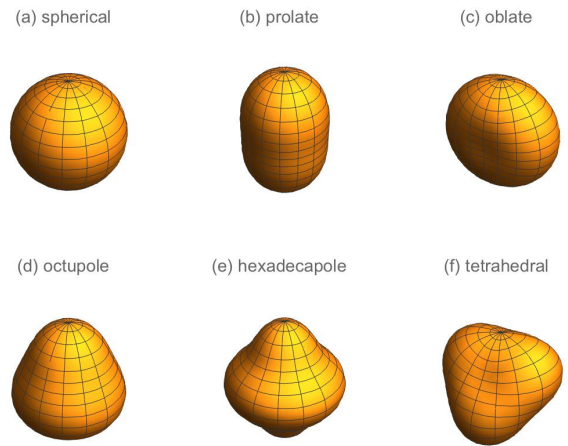


Fig. 1: Surfaces and their dependence on the variables  $\alpha_{\lambda\mu}$ . The cases shown are (a) spherical (all  $\alpha_{\lambda\mu}$  are zero); (b) prolate ( $\alpha_{20} > 0$ ); (c) oblate ( $\alpha_{20} < 0$ ); (d) octupole ( $\alpha_{30} \neq 0$ ) (e) hexadecapole ( $\alpha_{40} \neq 0$ ) and (f) tetrahedral (octupole with  $\alpha_{32} \neq 0$ ).

where  $\pi_{2\mu}$  is the momentum variable associated with  $\alpha_{2\mu}$ ,  $\pi_{2\mu} = B\dot{\alpha}_{2\mu}$ , with  $B$  the mass parameter and  $C$  the restoring force. The hamiltonian (4) corresponds to a five-dimensional harmonic oscillator in the collective variables  $\alpha_{2\mu}$  with frequency  $\omega = \sqrt{C/B}$  and vibrational energy  $\hbar\omega$ . After the introduction of the intrinsic coordinates ( $\beta, \gamma$ ) and the Euler angles, its quantization leads to the well-known Bohr–Mottelson hamiltonian,

$$\hat{H} = -\frac{\hbar^2}{2B} \left[ \frac{1}{\beta^4} \frac{\partial}{\partial \beta} \beta^4 \frac{\partial}{\partial \beta} + \frac{1}{\beta^2} \left( \frac{1}{\sin 3\gamma} \frac{\partial}{\partial \gamma} \sin 3\gamma \frac{\partial}{\partial \gamma} - \frac{1}{4} \sum_{\kappa} \frac{\hat{I}_{\kappa}^2}{\sin^2(\gamma - 2\pi\kappa/3)} \right) \right] + \beta^2,$$

where  $\hat{I}_{\kappa}$  are the components of the angular momentum operator in the intrinsic frame of reference. The solutions of this equation are known in complete detail [24, 25].

While quadrupole deformation in its various manifestations is prevalent throughout most of the periodic table, some evidence for octupole deformation can be found in the rare-earth and actinide regions. While this evidence usually exists in terms of vibrational oscillations, there are recent indications that the ground state of  $^{224}\text{Ra}$  has a permanent octupole deformation [26].

The geometric collective model and its extensions have been successful in describing a wide variety of nuclear properties. Commonly measured properties, including masses, angular momenta, magnetic moments and nuclear shapes, can be understood from the geometric collective model. Broad systematics of excited-state properties can likewise be described but with important input from microscopic considerations required.

## 2. Preamble: Wigner's supermultiplet model

In Wigner's supermultiplet model [27] nuclear forces are assumed to be invariant under rotations in spin as well as isospin space. A shell-model hamiltonian  $\hat{H}$  with this

property satisfies the commutation relations

$$[\hat{H}, \hat{S}_\mu] = [\hat{H}, \hat{T}_\nu] = [\hat{H}, \hat{Y}_{\mu\nu}] = 0,$$

where

$$\hat{S}_\mu = \sum_k \hat{s}_{k,\mu}, \quad \hat{T}_\nu = \sum_k \hat{t}_{k,\nu}, \quad \hat{Y}_{\mu\nu} = \sum_k \hat{s}_{k,\mu} \hat{t}_{k,\nu},$$

are the spin, isospin and spin–isospin operators in terms of  $\hat{s}_{k,\mu}$  and  $\hat{t}_{k,\nu}$ , the spin and isospin components of nucleon  $k$ . The set  $\{\hat{S}_\mu, \hat{T}_\nu, \hat{Y}_{\mu\nu}, \mu, \nu = 1, 2, 3\}$  generates the Lie algebra SU(4) and any hamiltonian that commutes with these 15 operators has SU(4) symmetry, in addition to the SU(2) symmetries associated with the total spin  $S$  and total isospin  $T$ .

The physical relevance of Wigner’s supermultiplet model follows from the short-range attractive nature of the nuclear interaction, which lowers the energy of states with increasing spatial symmetry. This principle can be given a precise mathematical formulation with the claim that the  $n$ -nucleon eigenstates of a nuclear hamiltonian with SU(4) symmetry are classified according to

$$\begin{array}{ccc} \text{U}(4\Lambda) & \supset & \text{U}_L(\Lambda) \otimes \text{U}_{ST}(4) \\ \downarrow & & \downarrow \quad \downarrow \\ [1^n] & & [\bar{h}] \quad [\bar{h}'] \end{array}, \quad (5)$$

where  $\Lambda$  is the orbital dimension of the (valence) single-particle space,  $\Lambda = \sum_\ell (2\ell + 1)$ , and  $4\Lambda$  is the total single-particle dimension for neutrons and protons, to account for the spin–isospin degrees of freedom. The labels underneath the algebras are explained below.

The order of the algebras  $\text{U}_L(4\Lambda)$  and  $\text{U}_L(\Lambda)$  (*i.e.*, their number of generators  $16\Lambda^2$  and  $\Lambda^2$ ) is determined by the orbital shells  $\ell$  that are included in the model space. It is not necessary to specify what orbital shells are considered but we assume in this section that summations over  $\ell$  consistently include all of them.

For the subsequent discussion it is convenient to use the formalism of second quantization and to introduce the operators  $a_{\ell m_\ell s m_s t m_t}^\dagger$  which create a nucleon in the orbital shell  $\ell$  with  $z$  projection  $m_\ell$ , spin  $s = \frac{1}{2}$  with  $z$  projection  $m_s$  and isospin  $t = \frac{1}{2}$  with  $z$  projection  $m_t$ . The corresponding annihilation operators are  $a_{\ell m_\ell s m_s t m_t}$  and, to ensure the correct transformation properties under rotations in orbital, spin and isospin space, one also introduces the modified annihilation operators  $\tilde{a}_{\ell m_\ell s m_s t m_t} \equiv (-)^{\ell+m_\ell+s+m_s+t+m_t} a_{\ell -m_\ell, s-m_s, t-m_t}$ . Anti-symmetry of the wave function is imposed and the Pauli principle is respected by requiring the following anti-commutation rules among the fermion creation and annihilation operators:

$$\begin{aligned} \{a_{\ell m_\ell s m_s t m_t}^\dagger, a_{\ell' m'_\ell s'_m s'_m t'_m t'_m}^\dagger\} &= \delta_{\ell\ell'} \delta_{m_\ell m'_\ell} \delta_{m_s m'_s} \delta_{m_t m'_t}, \\ \{a_{\ell m_\ell s m_s t m_t}, a_{\ell' m'_\ell s'_m s'_m t'_m t'_m}\} &= 0, \\ \{a_{\ell m_\ell s m_s t m_t}^\dagger, a_{\ell' m'_\ell s'_m s'_m t'_m t'_m}\} &= 0. \end{aligned} \quad (6)$$

The generators of the algebra  $\text{U}(4\Lambda)$  in equation (5) can be written in terms of the coupled tensors

$$(a_{\ell st}^\dagger \times \tilde{a}_{\ell' st})_{M_L M_S M_T}^{(LST)},$$

where the superscripts denote the coupling in orbital angular momentum  $L$ , spin  $S$  and isospin  $T$ , respectively, and

the subscripts refer to their respective projections  $M_L$ ,  $M_S$  and  $M_T$ . The explicit expression for the coupled tensors involves  $\text{SO}(3) \supset \text{SO}(2)$  coupling or standard Clebsch–Gordan coefficients [28],

$$\begin{aligned} &(a_{\ell st}^\dagger \times \tilde{a}_{\ell' st})_{M_L M_S M_T}^{(LST)} \\ &= \sum_{m_\ell m'_\ell} (\ell m_\ell \ell' m'_\ell | LM_L) \sum_{m_s m'_s} (s m_s s' m'_s | SM_S) \\ &\quad \times \sum_{m_t m'_t} (t m_t t' m'_t | TM_T) a_{\ell m_\ell s m_s t m_t}^\dagger \tilde{a}_{\ell' m'_\ell s' m'_s t' m'_t}. \end{aligned}$$

It is assumed that all physical operators (*e.g.*, the hamiltonian, electromagnetic transition operators, ...) can be written in terms of the generators of  $\text{U}_L(4\Lambda)$ . Algebras with this property are sometimes referred to as the dynamical or spectrum generating algebra of the system under study [29, 30].

The  $\ell s$ -coupled representation of nucleon creation and annihilation operators is the most convenient for the purposes of the present paper. Because of spin–orbit terms in the nucleon–nucleon interaction it is no longer used in present-day shell-model studies and is commonly replaced by a  $jj$ -coupled representation. Both representations are nevertheless equivalent as a result of the relation

$$a_{j m_j t m_t}^\dagger = \sum_{m_\ell m_s} (\ell m_\ell s m_s | j m_j) a_{\ell m_\ell s m_s t m_t}^\dagger,$$

where  $a_{j m_j t m_t}^\dagger$  creates a nucleon in the shell  $j$  with  $z$  projection  $m_j$  and with isospin  $t = \frac{1}{2}$  with  $z$  projection  $m_t$ . The corresponding annihilation operators are  $a_{j m_j t m_t}$  and the modified annihilation operators are defined as  $\tilde{a}_{j m_j t m_t} \equiv (-)^{j+m_j+t+m_t} a_{j -m_j, t-m_t}$ .

The generators of  $\text{U}_L(4\Lambda)$  can also be written in terms of the  $jj$ -coupled tensors

$$(a_{j t}^\dagger \times \tilde{a}_{j' t})_{M_J M_T}^{(JT)},$$

where the superscripts denote the coupling in total (*i.e.*,  $L$  plus  $S$ ) angular momentum  $J$  and isospin  $T$ , respectively, and the subscripts refer to their respective projections  $M_J$  and  $M_T$ . Whether one uses the  $\ell s$ - or  $jj$ -coupled representation is a matter of convenience since both sets of generators are related by the unitary transformation

$$\begin{aligned} (a_{\ell st}^\dagger \times \tilde{a}_{\ell' st})_{M_L M_S M_T}^{(LST)} &= \sum_{J M_J} (LM_L SM_S | JM_J) \\ &\quad \times \sum_{j j'} \begin{bmatrix} \ell & \frac{1}{2} & j \\ \ell' & \frac{1}{2} & j' \\ L & S & J \end{bmatrix} (a_{j t}^\dagger \times \tilde{a}_{j' t})_{M_J M_T}^{(JT)}, \end{aligned}$$

in terms of the unitary nine- $j$  symbol

$$\begin{bmatrix} j_1 & j_2 & J_{12} \\ j_3 & j_4 & J_{34} \\ J_{13} & J_{24} & J \end{bmatrix} \equiv \hat{J}_{12} \hat{J}_{34} \hat{J}_{13} \hat{J}_{24} \left\{ \begin{array}{ccc} j_1 & j_2 & J_{12} \\ j_3 & j_4 & J_{34} \\ J_{13} & J_{24} & J \end{array} \right\},$$

where  $\hat{x} \equiv \sqrt{2x+1}$  and the symbol in curly brackets is a standard nine- $j$  symbol [28].

Two subalgebras of  $\text{U}_L(4\Lambda)$  appear in equation (5). The first is  $\text{U}_L(\Lambda)$ , which has the generators

$$\hat{G}_\mu^{(\lambda)}(\ell\ell') \equiv (a_{\ell st}^\dagger \times \tilde{a}_{\ell' st})_{\mu 00}^{(\lambda 00)}, \quad (7)$$

that is, coupled tensors that are scalar in spin and isospin, as indicated by the superscripts  $S = T = 0$ . The angular momentum  $\lambda$  runs over all possible couplings of  $\ell$  and  $\ell'$  (which include all orbital shells of the model space),  $\lambda = |\ell - \ell'|, |\ell - \ell'| + 1, \dots, \ell + \ell'$ , and  $\mu$  is its projection,  $\mu = -\lambda, -\lambda + 1, \dots, +\lambda$ . The generators of  $U_L(\Lambda)$  can also be written in terms of the operators

$$\hat{G}_\mu^{(\lambda)}(jj') \equiv (a_{jt}^\dagger \times \tilde{a}_{j't})_{\mu 0}^{(\lambda 0)}. \quad (8)$$

These should not be confused with those in equation (7) since  $j$  and  $j'$  are always half-odd-integer, in contrast to the integer values of  $\ell$  and  $\ell'$ . The operators (8) generate the algebra  $U_L(2\Lambda)$ , which does not appear in the classification (5) but which contains the generators of  $U_L(\Lambda)$  as a subset since

$$\hat{G}_\mu^{(\lambda)}(\ell\ell') = \frac{(-)^{\ell+j'+1/2+\lambda}}{\sqrt{2}} \hat{j}\hat{j}' \left\{ \begin{matrix} j & \ell & \frac{1}{2} \\ \ell' & j' & \lambda \end{matrix} \right\} \hat{G}_\mu^{(\lambda)}(jj'),$$

where the symbol in curly brackets is a six- $j$  symbol [28]. This relation allows us to write the  $\ell s$ -coupled tensors  $\hat{G}_\mu^{(\lambda)}(\ell\ell')$  in terms of the  $jj$ -coupled tensors  $\hat{G}_\mu^{(\lambda)}(jj')$ , which is more convenient for present-day applications.

The second subalgebra of  $U_L(4\Lambda)$  in equation (5) is  $U_{ST}(4)$  with the generators

$$\sum_\ell (a_{\ell st}^\dagger \times \tilde{a}_{\ell st})_{0M_S M_T}^{(0ST)},$$

which are scalar in orbital space and with spin  $S$  and isospin  $T$  equal to 0 or 1. The explicit definition of the  $U_{ST}(4)$  generators is

$$\begin{aligned} \hat{n} &\equiv 2 \sum_\ell \sqrt{2\ell+1} (a_{\ell st}^\dagger \times \tilde{a}_{\ell st})_{000}^{(000)}, \\ \hat{S}_\mu &\equiv \sum_\ell \sqrt{2(2\ell+1)} (a_{\ell st}^\dagger \times \tilde{a}_{\ell st})_{0\mu 0}^{(010)}, \\ \hat{T}_\nu &\equiv \sum_\ell \sqrt{2(2\ell+1)} (a_{\ell st}^\dagger \times \tilde{a}_{\ell st})_{00\nu}^{(001)}, \\ \hat{Y}_{\mu\nu} &\equiv \sum_\ell \sqrt{2\ell+1} (a_{\ell st}^\dagger \times \tilde{a}_{\ell st})_{0\mu\nu}^{(011)}, \end{aligned} \quad (9)$$

corresponding to the number, spin, isospin and spin-isospin operators written in second quantization.

It can be shown that the generators of  $U_L(\Lambda)$  and those of  $U_{ST}(4)$  close under commutation, and that they commute with each other. In particular, the following commutator property among the generators of  $U_L(\Lambda)$  is valid:

$$\begin{aligned} [\hat{G}_\mu^{(\lambda)}(\ell_1\ell_2), \hat{G}_{\mu'}^{(\lambda')}(\ell_3\ell_4)] &= \frac{1}{2} \hat{\lambda}\hat{\lambda}' \sum_{\lambda''\mu''} (\lambda\mu \lambda'\mu' | \lambda''\mu'') \\ &\times \left[ (-)^{\lambda'+\ell_1+\ell_4} \left\{ \begin{matrix} \lambda & \lambda' & \lambda'' \\ \ell_4 & \ell_1 & \ell_2 \end{matrix} \right\} \delta_{\ell_2\ell_3} \hat{G}_{\mu''}^{(\lambda'')}(\ell_1\ell_4) \right. \\ &\left. - (-)^{\lambda+\lambda'+\ell_2+\ell_3} \left\{ \begin{matrix} \lambda & \lambda' & \lambda'' \\ \ell_3 & \ell_2 & \ell_1 \end{matrix} \right\} \delta_{\ell_1\ell_4} \hat{G}_{\mu''}^{(\lambda'')}(\ell_3\ell_2) \right]. \end{aligned} \quad (10)$$

The (tedious) derivation of equation (10) requires the expansion of the coupled tensors (7) in terms of uncoupled generators. It makes use of the anti-commutators (6) and of summation properties of Clebsch–Gordan coefficients and six- $j$  symbols [28]. The relation (10) is identical to

the corresponding one for bosons [31] but for the factor  $1/2$  which originates from the coupling in spin and isospin. The relation is central to the subsequent discussion since many properties concerning various orbital classifications can be derived from it.

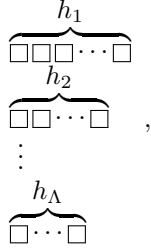
Algebraic models in nuclear physics—and generally in quantum physics—rely extensively on the notion of irreducible representation, which is used to label eigenstates of hamiltonians with certain symmetries. Because a system of  $n$  identical particles is invariant under permutations that exchange all coordinates of any two particles, representations of the permutation group  $S_n$ , consisting of all permutations of  $n$  objects, are of central importance. As the system's hamiltonian is invariant under  $S_n$ , its eigenstates are labelled by the irreducible representations of  $S_n$ . If no condition is imposed other than invariance under the exchange of *all* coordinates, nothing more can be learned from permutation symmetry than the fact that any nuclear eigenstate must be anti-symmetric. The classification (5), however, imposes the invariance under the exchange of *only* the spatial (or, equivalently, *only* the spin-isospin) coordinates of any two particles. The symmetry character under such *partial* permutations can be exploited to yield additional quantum numbers.

The symmetry type under a (total or partial) permutation of a system of  $n$  particles that occupy  $\Lambda$  single-particle states is characterized by a set of integers that satisfy the conditions  $h_1 \geq h_2 \geq \dots \geq h_\Lambda \geq 0$  and  $h_1 + h_2 + \dots + h_\Lambda = n$ . This set of integers is denoted here as  $[\bar{h}] \equiv [h_1, \dots, h_\Lambda]$  and is often represented as a Young pattern or diagram, which corresponds to  $n$  boxes that are placed in  $\Lambda$  rows of length  $h_1, h_2, \dots$ , one underneath the other, beginning with  $h_1$ . An irreducible representation of  $S_n$  characterized by a given Young diagram contains basis states that are obtained by placing each of the  $n$  particles in a box according to the following rule. All particles are given a label between 1 and  $n$ . They are then distributed over the boxes such that in each row the particle index increases from left to right and in each column the index increases from top to bottom. The basis states obtained in this way are called Young tableaux. For a given Young diagram each Young tableau corresponds to a different state with a given mixed symmetry, which is obtained by anti-symmetrization in the particles belonging to the same column, after symmetrization in the particles belonging to the same row (or *vice versa*). In the case of complete anti-symmetry the Young diagram reduces to a single column of  $n$  boxes ( $n \leq \Lambda$  because of the Pauli principle) with only a single associated Young tableau, namely the one with increasing particle index from top to bottom. This shows that in the case of overall anti-symmetry all states have an identical permutational character.

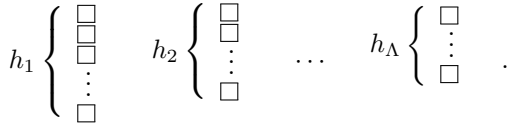
The mathematical theory of group representations is developed in the monographs by Murnaghan [32] and Littlewood [33] while its applications to physical problems are described in the treatise of Hamermesh [34]. A clear and succinct account of the use of Young diagrams in many-body quantum physics is given by Lipas [35].

The states (5) are characterized by a certain symmetry  $[\bar{h}] \equiv [h_1, \dots, h_\Lambda]$  in orbital space and a concomitant spin-isospin symmetry  $[\bar{h}'] \equiv [h'_1, h'_2, h'_3, h'_4]$ , with  $h_1 + \dots + h_\Lambda = h'_1 + h'_2 + h'_3 + h'_4 = n$ , the total number of

nucleons in the valence space. The symmetry type under the partial exchange of coordinates (*e.g.*, the exchange of only the spatial or only the spin–isospin coordinates) is thus characterized by a Young diagram. To ensure the overall anti-symmetry of the wave function, which is indicated in equation (5) by the one-column Young diagram  $[1^n] \equiv [1, 1, \dots, 1]$  of  $U_L(4\Lambda)$ , the two Young diagrams  $[\bar{h}]$  and  $[\bar{h}']$  must be conjugate, that is, they are obtained from each other by interchanging rows and columns. The conjugate relation is most easily explained with a figure: A general Young diagram



becomes after conjugation



Since the spin–isospin algebra  $U_{ST}(4)$  is characterized by a Young diagram of at most four rows, it follows that the Young diagram associated with the orbital algebra  $U_L(\Lambda)$  has no more than four columns or, equivalently, has rows with at most four boxes,  $4 \geq h_1 \geq h_2 \geq \dots \geq h_\Lambda \geq 0$ . This is the group-theoretical transcription of the fact that, in an anti-symmetric many-body wave function, the same orbital single-particle state cannot be occupied by more than four particles corresponding to the four different nucleonic intrinsic states with spin and isospin up or down.

The number operator  $\hat{n}$  is common to both  $U_L(\Lambda)$  and  $U_{ST}(4)$ , and therefore  $U_L(\Lambda) \otimes U_{ST}(4)$  is in fact not a direct product as the algebras involved are not disjoint. This can be easily remedied by considering the direct product  $U_L(\Lambda) \otimes SU_{ST}(4)$ , where the number operator is dropped from  $U_{ST}(4)$  to give  $SU_{ST}(4)$ . Irreducible representations of the latter algebra are characterized by three labels, which can be chosen as  $[h'_1 - h'_4, h'_2 - h'_4, h'_3 - h'_4]$  or, alternatively, as  $(\lambda', \mu', \nu')$  with

$$\lambda' \equiv h'_1 - h'_2, \quad \mu' \equiv h'_2 - h'_3, \quad \nu' \equiv h'_3 - h'_4,$$

which corresponds to the more conventional notation of the supermultiplet model. We conclude therefore that the classification (5) can be replaced by an equivalent one which reads

$$\begin{array}{ccc} U(4\Lambda) & \supset & U_L(\Lambda) \otimes SU_{ST}(4) \\ \downarrow & & \downarrow \quad \downarrow \\ [1^n] & & [\bar{h}] \quad (\lambda', \mu', \nu') \end{array}, \quad (11)$$

where it is understood that the labels  $(\lambda', \mu', \nu')$  are derived from the  $U_{ST}(4)$  labels  $[\bar{h}']$ , which correspond to a Young diagram that is conjugate to  $[\bar{h}]$ . In the following we use the classification (5) or (11), whichever is most convenient, with the understanding that both are equivalent.

A ‘supermultiplet’ consists of all states contained in an irreducible representation  $[\bar{h}]$  of the orbital algebra  $U_L(\Lambda)$  or, equivalently, in an irreducible representation  $(\lambda', \mu', \nu')$  of the spin–isospin algebra  $SU_{ST}(4)$ . The central idea of Wigner’s supermultiplet model is that the nucleon–nucleon interaction strongly favours states with maximal spatial symmetry and that as a consequence different supermultiplets are well separated in energy. Low-energy states in the spectrum of a given nucleus have maximal spatial symmetry and therefore belong to the so-called ‘favoured supermultiplet’.

The separation of supermultiplets can be achieved by an interaction of the form

$$\hat{C}_2[U_L(\Lambda)] = 4 \sum_{\ell\ell'} \sum_{\lambda} (-)^{\ell+\ell'} \hat{G}^{(\lambda)}(\ell\ell') \cdot \hat{G}^{(\lambda)}(\ell\ell'), \quad (12)$$

where the dot denotes a scalar product,

$$\hat{T}^{(\lambda)} \cdot \hat{T}^{(\lambda)} \equiv (-)^{\lambda} \sqrt{2\lambda+1} (\hat{T}^{(\lambda)} \times \hat{T}^{(\lambda)})_0^{(0)}.$$

The operator  $\hat{C}_2[U_L(\Lambda)]$  commutes with all generators of  $U_L(\Lambda)$ ,

$$[\hat{G}_\mu^{(\lambda)}(\ell\ell'), \hat{C}_2[U_L(\Lambda)]] = 0,$$

and therefore can be associated with the quadratic Casimir operator of  $U_L(\Lambda)$ , as anticipated by the notation. To prove this commutator property use is made of equation (10) together with the operator identity

$$[\hat{A}, \hat{B}\hat{C}] = \hat{B}[\hat{A}, \hat{C}] + [\hat{A}, \hat{B}]\hat{C}.$$

The proof is straightforward, to the extent that it can be delivered, for particular realizations of  $U_L(\Lambda)$ , in a symbolic language like **Mathematica** [36]. The presence of the phase factor  $(-)^{\ell+\ell'}$  in the expression (12) should be noted; it has no importance if orbital shells of a single oscillator shell are included but matters in case of orbital shells with both parities.

All bases considered in this paper have the reduction (5) or (11) as a starting point. Since the operator  $\hat{C}_2[U_L(\Lambda)]$  commutes with all generators of  $U_L(\Lambda)$ , its eigenvalues in such bases are known from classical group theory (see, for example, table 5.1 of Ref. [31]) to be

$$\sum_{i=1}^{\Lambda} h_i (h_i + \Lambda + 1 - 2i), \quad (13)$$

where  $h_i$  are the labels associated with  $U_L(\Lambda)$ . Casimir operators are only determined up to a proportionality factor and the coefficient ‘4’ in equation (12) is chosen such that the expectation value of  $\hat{C}_2[U_L(\Lambda)]$  yields the eigenvalue (13).

As the representations  $[\bar{h}]$  and  $[\bar{h}']$  are conjugate, an entirely equivalent interaction can be proposed in terms of the quadratic Casimir operator of the spin–isospin algebra  $U_{ST}(4)$ . The operator  $\hat{C}_2[U_{ST}(4)]$  is also diagonal in any basis associated with equation (5) or (11) and its eigenvalues are

$$h'_1(h'_1 + 3) + h'_2(h'_2 + 1) + h'_3(h'_3 - 1) + h'_4(h'_4 - 3),$$

where  $h'_i$  are the labels associated with  $U_{ST}(4)$ . Given that  $[\bar{h}]$  and  $[\bar{h}']$  are conjugate Young diagrams, the following relation between the two Casimir operators can be established:

$$\hat{C}_2[U_L(\Lambda)] = (\Lambda + 4)\hat{n} - \hat{C}_2[U_{ST}(4)].$$

We emphasize that this relation is *not* generally valid but that it applies only in the anti-symmetric representation  $[1^n]$  of  $U(4\Lambda)$  since it uses the fact that  $[\bar{h}]$  and  $[\bar{h}']$  are conjugate.

For completeness we also quote the expression for the quadratic Casimir operator of  $SU_{ST}(4)$ . In terms of the labels  $(\lambda', \mu', \nu')$  the eigenvalue of  $4\hat{C}_2[U_{ST}(4)]$  is rewritten as

$$3\lambda'(\lambda'+4)+4\mu'(\mu'+4)+3\nu'(\nu'+4)+4\mu'(\lambda'+\nu')+2\lambda'\nu'+n^2.$$

Therefore, since  $n$  is a constant for a given nucleus, we may define

$$\hat{C}_2[SU_{ST}(4)] \equiv 4\hat{C}_2[U_{ST}(4)] - \hat{n}^2,$$

and we find that the Casimir operator  $\hat{C}_2[SU_{ST}(4)]$  has the eigenvalues

$$3\lambda'(\lambda'+4)+4\mu'(\mu'+4)+3\nu'(\nu'+4)+4\mu'(\lambda'+\nu')+2\lambda'\nu', \quad (14)$$

which corresponds to the conventional expression of the supermultiplet model, as quoted for example in Ref. [38].

States with maximal spatial symmetry have the largest possible eigenvalue of  $\hat{C}_2[U_L(\Lambda)]$  or, equivalently, the smallest possible eigenvalue of  $\hat{C}_2[U_{ST}(4)]$  or  $\hat{C}_2[SU_{ST}(4)]$ . For a given nucleus with  $N$  neutrons and  $Z$  protons, and isospin projection  $T_z = (N - Z)/2$ , the favoured supermultiplet must be compatible with the minimal isospin  $T = |T_z|$  of states in that nucleus. The allowed values of the total spin  $S$  and the total isospin  $T$  in a given supermultiplet  $(\lambda', \mu', \nu')$  are found from the branching rule associated with

$$\begin{array}{ccc} SU_{ST}(4) & \supset & SU_S(2) \otimes SU_T(2) \\ \downarrow & & \downarrow \quad \downarrow \\ (\lambda', \mu', \nu') & & S \quad T \end{array} . \quad (15)$$

Most cases of interest for this branching rule have been tabulated (see, for example, Refs. [37, 38]). All cases can be found starting from the equivalent branching rule for

$$\begin{array}{ccc} U_{ST}(4) & \supset & U_S(2) \otimes U_T(2) \\ \downarrow & & \downarrow \quad \downarrow \\ [\bar{h}'] & & [s_1, s_2] \quad [t_1, t_2] \end{array} ,$$

with the relation between  $(\lambda', \mu', \nu')$  and  $[\bar{h}']$  as explained above, and with  $S = (s_1 - s_2)/2$  and  $T = (t_1 - t_2)/2$ . This is a particular case of the branching rule for  $U(n_1 n_2) \supset U(n_1) \otimes U(n_2)$ , the algorithm of which is explained in the appendix.

The favoured supermultiplet for a given nucleus is therefore found from the following procedure:

- Determine the number  $n$  of neutrons and protons in the valence shell.
- For that value of  $n$ , enumerate all possible Young diagrams  $[\bar{h}]$  associated with  $U_L(\Lambda)$ . This amounts to finding all partitions of  $n$  into  $\Lambda$  integers  $h_1, h_2, \dots, h_\Lambda$  that satisfy  $4 \geq h_1 \geq h_2 \geq \dots \geq h_\Lambda \geq 0$ .
- For each  $[\bar{h}]$  find the conjugate Young diagram  $[\bar{h}']$ , the corresponding labels  $(\lambda', \mu', \nu')$  and the allowed values of  $S$  and  $T$  from the branching rule (15).

- The favoured supermultiplet corresponds to the  $SU_{ST}(4)$  irreducible representation  $(\lambda', \mu', \nu')$  with the smallest possible eigenvalue (14) which contains the isospin  $T = |T_z|$ .

Through the application of the above procedure, generic rules emerge that determine the favoured supermultiplet for any nucleus, depending on whether it is even–even, odd–mass or odd–odd. From the summary shown in table 1 it is seen that the  $U_{ST}(4)$  labels  $[\bar{h}']$  are functions of the number of valence nucleons  $n$  and the isospin projection  $|T_z|$  while the  $SU_{ST}(4)$  labels  $(\lambda', \mu', \nu')$  more conveniently only depend on  $|T_z|$ . Once  $[\bar{h}']$  is determined, the orbital symmetry  $[\bar{h}]$  follows from conjugation.

We conclude the discussion of Wigner’s supermultiplet model with two comments concerning  $N = Z$  nuclei. The above procedure does not *a priori* exclude that an isospin larger than  $|T_z|$  is also contained in the favoured supermultiplet. With one exception this in fact never happens. In other words, states with  $T > |T_z|$  almost always belong to the next-favoured supermultiplet and, on account of this finding, are well separated in energy from states with  $T = |T_z|$ . The one exception concerns odd–odd  $N = Z$  nuclei: According to table 1 the favoured supermultiplet in that case is  $(\lambda', \mu', \nu') = (0, 1, 0)$ , which contains both  $T = 0$  and  $T = 1$ . Low-energy levels in odd–odd  $N = Z$  nuclei, uniquely, indeed carry both isospins. This empirical observation, the explanation of which traditionally invokes convoluted arguments about counterbalancing effects of pairing and symmetry energies [39, 40], can be explained naturally in the framework of the  $SU(4)$  model.

The second comment concerns the so-called Wigner binding energy. Self-conjugate nuclei, that is, nuclei with an equal number of neutrons and protons ( $N = Z$ ), are unusually tightly bound. This extra binding energy was first noted by Wigner who proposed an explanation of the observed “kinks in the mass defect curve” with symmetry arguments [41]. Since then, the  $N = Z$  cusp in the nuclear mass surface is often described with an additional term in binding-energy formulas, known as the Wigner binding energy.

The Wigner binding energy  $B_W(N, Z)$  consists of two parts [42]

$$B_W(N, Z) = -W(A)|N - Z| - d(A)\delta_{N,Z}\pi_{np},$$

where  $W(A)$  and  $d(A)$  are functions of the atomic mass number  $A \equiv N + Z$ . The first term on the right-hand side gives rise to a cusp in the binding energy at  $N = Z$ . By definition, the quantity  $\pi_{np}$  equals 1 for odd–odd nuclei and vanishes otherwise, and therefore  $d(A)$  matters only for odd–odd  $N = Z$  nuclei. Empirical estimates lead to  $W(A) \approx 47A^{-1}$  MeV and  $d(A)$  comparable but possibly somewhat smaller [43].

The origin of the two terms in the Wigner binding energy can be explained on the basis of previously derived results. Insertion of the favoured supermultiplet labels  $(\lambda', \mu', \nu')$ , as given in table 1, into the eigenvalue (14) of  $\hat{C}_2[SU_{ST}(4)]$  leads to the expression [44]

$$(N - Z)^2 + 8|N - Z| + 8\delta_{N,Z}\pi_{np} + 6\delta_p(N, Z).$$

One recovers the classical symmetry energy of nuclear mass formulas in the first term. The last term has a



Table 1: Favoured SU(4) supermultiplets of nuclei with  $n$  valence nucleons.

Nucleus	$[\bar{h}]^a$	$(\lambda', \mu', \nu')$
Even-even	$[k +  T_z , k +  T_z , k, k]$	$(0,  T_z , 0)$
Odd-mass	$[k +  T_z  + \frac{1}{2}, k +  T_z  - \frac{1}{2}, k, k]$	$(1,  T_z  - \frac{1}{2}, 0)$
Odd-odd ( $N \neq Z$ )	$[k' +  T_z  + 1, k' +  T_z , k' + 1, k']$	$(1,  T_z  - 1, 1)$
Odd-odd ( $N = Z$ )	$[k' + 1, k' + 1, k', k']$	$(0, 1, 0)$

<sup>a</sup>With  $k = (n - 2|T_z|)/4$ ;  $k' = (n - 2|T_z| - 2)/4$ .

pairing-like character and follows the somewhat unusual convention that  $\delta_p(N, Z)$  is 0 for even-even, 1 for odd-mass and 2 for odd-odd nuclei. The second and third terms correspond exactly to those in  $B_W(N, Z)$  with the constraint  $W(A) = d(A) = 8g(A)$ , where  $g(A)$  is the coefficient in front of  $\hat{C}_2[\text{SU}_{ST}(4)]$ . (The change in sign follows from the fact that the eigenvalue of  $\hat{C}_2[\text{SU}_{ST}(4)]$  refers to an *interaction* energy between nucleons whereas  $B_W(N, Z)$  is a *binding* energy.) We therefore conclude that the Wigner term  $B_W(N, Z)$  in nuclear mass formulas is directly inspired by the supermultiplet model.

Wigner's supermultiplet model is based on an  $LS$ -coupling scheme which is now known to be inappropriate for most nuclei. The breakdown of SU(4) symmetry is a consequence of the spin- and isospin-dependent (Coulomb) interactions, in particular the spin-orbit term in the nuclear mean field. Nevertheless, Wigner's idea remains relevant because it illustrates the connection between the short-range character of the nuclear interaction and the spatial symmetry of the many-body wave function. One of the most important ramifications of the supermultiplet model is its extension to include rotational motion in the spherical shell model by way of Elliott's SU(3) model.

### 3. Elliott's SU(3) model of quadrupole deformation

In this section we review Elliott's SU(3) model [45], which can be considered as a further elaboration upon the supermultiplet scheme. A convenient way to understand this symmetry takes the isotropic harmonic oscillator as a starting point, whose degeneracies can be interpreted in terms of a U(3) algebra [46]. A natural realization of this algebra is in a cartesian basis but, as shown by Elliott [45], an equivalent representation is possible in a spherical basis and leads to a realization of U(3) in terms of a number, an angular momentum and a quadrupole operator.

The subsequent presentation of the SU(3) model differs somewhat from that in Elliott's original papers in the sense that we assume, as is customary in present-day shell-model calculations, a given single-particle valence space, in which the many-body nuclear hamiltonian is diagonalized. From this point of view, Elliott's result can be succinctly summarized as follows. If the single-particle space consists of one or several entire, degenerate oscillator shells and if the interaction between the nucleons in that space is exclusively of the quadrupole type, then an SU(3) dynamical symmetry results. We first present the analysis for one oscillator shell and then discuss extensions to more shells.

#### 3.1. One oscillator shell

For nucleons occupying an entire shell of the harmonic oscillator in three dimensions, with major quantum number  $N$  and orbital shells  $\ell = N, N-2, \dots, 1$  or 0, the following generic orbital classification can be proposed:

$$\begin{array}{ccccccc} \text{U}(\Gamma) & \supset & \text{U}(3) & \supset & \text{SU}(3) & \supset & \text{SO}(3) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ [\bar{h}] & & [\bar{h}'] & & (\lambda, \mu) & & K \quad L \end{array}, \quad (16)$$

where the subscript ' $L$ ' is omitted from the orbital algebras for simplicity's sake and the notation  $\Gamma \equiv (N+1)(N+2)/2$  is introduced. The algebra U(3) consists of the number operator  $\hat{n}$  [see equation (9)], the three components of the angular momentum operator,

$$\hat{L}_\mu \equiv \sum_\ell \sqrt{\frac{4\ell(\ell+1)(2\ell+1)}{3}} \hat{G}_\mu^{(1)}(\ell\ell), \quad (17)$$

and the five components of the quadrupole operator,

$$\hat{Q}_\mu \equiv \sqrt{\frac{64\pi}{5}} \sum_{\ell\ell'} \frac{\langle \ell || r^2 Y_2 || \ell' \rangle}{\sqrt{5}} \hat{G}_\mu^{(2)}(\ell\ell'), \quad (18)$$

where it is assumed that the summations are over  $\ell, \ell' = N, N-2, \dots, 1$  or 0. The algebra SU(3) in the orbital classification (16) consists of  $\hat{L}_\mu$  and  $\hat{Q}_\mu$ . The angular momentum operator  $\hat{L}_\mu$  has a fixed structure, independent of the potential well, while the expression for the quadrupole operator  $\hat{Q}_\mu$  involves the reduced matrix elements  $\langle \ell || r^2 Y_2 || \ell' \rangle$  that do depend on radial integrals. These are well known for a harmonic oscillator [28],

$$\begin{aligned} \langle \ell || r^2 Y_2 || \ell \rangle &= -(2N+3) \sqrt{\frac{5}{16\pi} \frac{\ell(\ell+1)(2\ell+1)}{(2\ell-1)(2\ell+3)}}, \\ \langle \ell || r^2 Y_2 || \ell+2 \rangle &= -\sqrt{\frac{5}{16\pi} \frac{6(\ell+1)(\ell+2)(N-\ell)(N+\ell+3)}{2\ell+3}}. \end{aligned}$$

It was shown by Elliott [45] that the set of eight operators  $\{\hat{L}_\mu, \hat{Q}_\mu\}$ , pertaining to the harmonic oscillator, close under commutation and hence form a subalgebra of U(3). The commutation relations

$$\begin{aligned} [\hat{L}_\mu, \hat{L}_\nu] &= -\sqrt{2} (1\mu 1\nu | 1\mu + \nu) \hat{L}_{\mu+\nu}, \\ [\hat{L}_\mu, \hat{Q}_\nu] &= -\sqrt{6} (1\mu 2\nu | 2\mu + \nu) \hat{Q}_{\mu+\nu}, \\ [\hat{Q}_\mu, \hat{Q}_\nu] &= 3\sqrt{10} (2\mu 2\nu | 1\mu + \nu) \hat{L}_{\mu+\nu}, \end{aligned} \quad (19)$$

follow from the straightforward application of the commutator property (10) and can be proven, for particular

realizations of  $U(\Gamma)$  (*i.e.*, for particular values of  $N$ ), in a symbolic language like `Mathematica` [36].

In the orbital classification (16) appear the  $U(\Gamma)$  labels  $[\bar{h}]$  that are carried over from Wigner's supermultiplet model. The algebra  $U(3)$  is characterized by a three-rowed Young diagram  $[\bar{h}''] \equiv [h''_1, h''_2, h''_3]$  while the conventional notation for the  $SU(3)$  labels is  $(\lambda, \mu)$  with

$$\lambda \equiv h''_1 - h''_2, \quad \mu \equiv h''_2 - h''_3.$$

The label  $L$  in equation (16) refers to the total orbital angular momentum of the nucleons and an additional index  $K$  occurs not associated with any algebra but needed as a multiplicity label in the branching rule for  $SU(3) \supset SO(3)$ . The allowed Young diagrams  $[\bar{h}'']$  of  $U(3)$  or, equivalently, the allowed  $SU(3)$  labels  $(\lambda, \mu)$  for a given Young diagram  $[\bar{h}]$  of  $U(\Gamma)$  are determined from the branching rule for  $U(\Gamma) \supset U(3)$ , for which a general method exists known as the plethysm of  $S$  functions [33]. Furthermore, the branching rule for  $SU(3) \supset SO(3)$  can be obtained from  $S$ -function theory for restricted (*i.e.*, symplectic or orthogonal) algebras. This determines the allowed values of  $L$  in a given irreducible representation  $(\lambda, \mu)$ .

Group-theoretical methods related to  $S$ -function theory are described, for example, in the book by Wybourne [47]. A succinct summary of these methods, which suffices for the applications discussed in this paper, is given in the appendix.

Rather than giving the actual branching rules, which can be found in the original papers by Elliott [45], we note that each irreducible representation  $(\lambda, \mu)$  contains the orbital angular momenta  $L$  typical of a rotational band, cut off at some upper limit. The label  $K$  defines the intrinsic state associated to that band and can be interpreted as the projection of the orbital angular momentum  $L$  on the axis of symmetry of the rotating deformed nucleus.

The combination  $2\hat{n}^2 + 3\hat{L} \cdot \hat{L} + \hat{Q} \cdot \hat{Q}$  commutes with all generators of  $U(3)$  and hence can be identified with the quadratic Casimir operator of  $U(3)$ ,

$$\hat{C}_2[U(3)] = \frac{1}{3}\hat{n}^2 + \frac{1}{2}\hat{L} \cdot \hat{L} + \frac{1}{6}\hat{Q} \cdot \hat{Q},$$

which according to equation (13) has the eigenvalues

$$h''_1(h''_1 + 2) + h''_2(h''_2 + 1) + h''_3(h''_3 - 2).$$

In terms of the labels  $(\lambda, \mu)$  of  $SU(3)$  the eigenvalue of  $3\hat{C}_2[U(3)]$  is rewritten as

$$2\lambda(\lambda + 3) + 2\mu(\mu + 3) + 2\lambda\mu + n^2.$$

Therefore, since  $n$  is a constant for a given nucleus, we may define

$$\hat{C}_2[SU(3)] = 3\hat{C}_2[U(3)] - \hat{n}^2 = \frac{3}{2}\hat{L} \cdot \hat{L} + \frac{1}{2}\hat{Q} \cdot \hat{Q}, \quad (20)$$

which has the eigenvalues

$$2\lambda(\lambda + 3) + 2\mu(\mu + 3) + 2\lambda\mu. \quad (21)$$

The quadrupole interaction is thus a combination of Casimir operators,

$$-\hat{Q} \cdot \hat{Q} = -2\hat{C}_2[SU(3)] + 3\hat{C}_2[SO(3)]. \quad (22)$$

Since this hamiltonian can be written as a combination of Casimir operators belonging to the chain (16) of nested algebras, it is solvable with eigenstates

$$|[1^n]; [\bar{h}](\lambda, \mu)KL \times [\bar{h}']ST\rangle, \quad (23)$$

and energy eigenvalues

$$-4\lambda(\lambda + 3) - 4\mu(\mu + 3) - 4\lambda\mu + 3L(L + 1). \quad (24)$$

We illustrate the procedure to obtain the complete eigenspectrum of the Casimir operator (20) with the example of four nucleons in the  $sd$  shell (*i.e.*,  $n = 4$  and  $\Gamma = 6$ ). The allowed  $U(6)$  labels  $[\bar{h}]$  correspond to all partitions of  $n = 4$  and they are thus given by

$$[\bar{h}] = [4], [3, 1], [2, 2], [2, 1, 1], [1, 1, 1, 1].$$

The equivalent series of  $U(4)$  labels consists of the conjugate Young diagrams,

$$[\bar{h}'] = [1, 1, 1, 1], [2, 1, 1], [2, 2], [3, 1], [4].$$

The former series determines the allowed  $SU(3)$  labels  $(\lambda, \mu)$ , which follow from the branching rules for  $U(6) \supset SU(3)$  and can be taken from Elliott [45] (see also the appendix),

$$\begin{aligned} [4] &\mapsto (8, 0) + (4, 2) + (0, 4) + (2, 0), \\ [3, 1] &\mapsto (6, 1) + (4, 2) + (2, 3) + (3, 1) + (1, 2) + (2, 0), \\ [2^2] &\mapsto (4, 2) + (0, 4) + (3, 1) + (2, 0), \\ [2, 1^2] &\mapsto (5, 0) + (2, 3) + (3, 1) + (1, 2) + (0, 1), \\ [1^4] &\mapsto (1, 2), \end{aligned}$$

while the latter series determines the allowed  $(S, T)$  values, which follow from the branching rules for  $U(4) \supset SU(2) \times SU(2)$  (see Ref. [37] and also the appendix),

$$\begin{aligned} [1^4] &\mapsto (0, 0), \\ [2, 1^2] &\mapsto (0, 1) + (1, 0) + (1, 1), \\ [2^2] &\mapsto (0, 0) + (1, 1) + (0, 2) + (2, 0), \\ [3, 1] &\mapsto (0, 1) + (1, 0) + (1, 1) + (1, 2) + (2, 1), \\ [4] &\mapsto (0, 0) + (1, 1) + (2, 2). \end{aligned}$$

By combining the information from both types of branching rule, we know therefore for each of the nine possible  $(S, T)$  combinations what are the allowed  $SU(3)$  labels and how many times they occur.

The resulting eigenspectrum is displayed in figure 2. The spectrum in the left panel is complete and shows all irreducible representations  $(\lambda, \mu)$  with their multiplicities in the five possible supermultiplets. As argued in section 2, the nuclear interaction lowers the energy of the states with maximal spatial symmetry that occur in the favoured supermultiplet. The latter depends on the isospin projection  $T_z$  or, equivalently, on the nucleon numbers  $N$  and  $Z$ , since it limits the allowed values of isospin through  $T \geq |T_z|$ . For two neutrons and two protons,  $T_z = 0$ , the favoured supermultiplet is  $[\bar{h}] = [4]$  of  $U_L(6)$ , which corresponds to  $[\bar{h}'] = [1, 1, 1, 1]$  of  $U_{ST}(4)$  or to  $(\lambda', \mu', \nu') = (0, 0, 0)$  of  $SU_{ST}(4)$ . For three neutrons and one proton or its mirror system,  $T_z = \pm 1$ , the favoured supermultiplet is

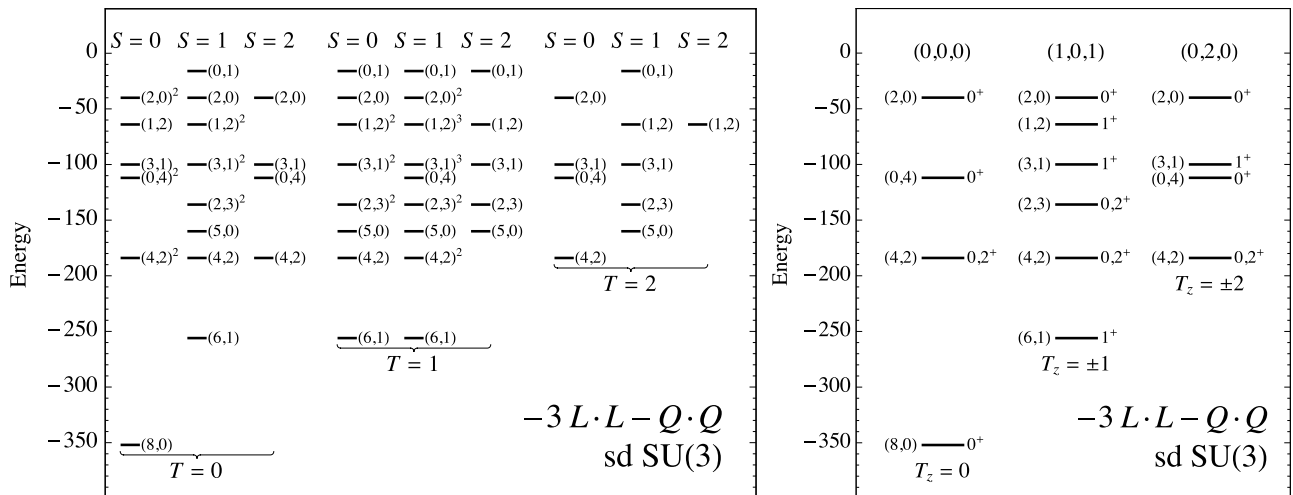


Fig. 2: The eigenspectrum of the operator  $-3\hat{L} \cdot \hat{L} - \hat{Q} \cdot \hat{Q}$  for four nucleons in the *sd* shell. Left panel: Levels are labelled by the SU(3) quantum numbers  $(\lambda, \mu)$  and values of the total spin  $S$  and the total isospin  $T$  are also indicated. If an irreducible representation  $(\lambda, \mu)$  occurs  $k$  times for a given  $S$  and  $T$ , this is indicated by a superscript as  $(\lambda, \mu)^k$ . Right panel: Only levels in the favoured supermultiplets are shown, which are  $(0, 0, 0)$  for  $T_z = 0$ ,  $(1, 0, 1)$  for  $T_z = \pm 1$  and  $(0, 2, 0)$  for  $T_z = \pm 2$ . Levels are labelled by the SU(3) quantum numbers  $(\lambda, \mu)$  on the left and by the projections  $K$  and parity  $\pi = +$  on the right. The supermultiplet labels  $(\lambda', \mu', \nu')$  and the isospin projection  $T_z$  are also indicated, and all levels have  $S = 0$ .

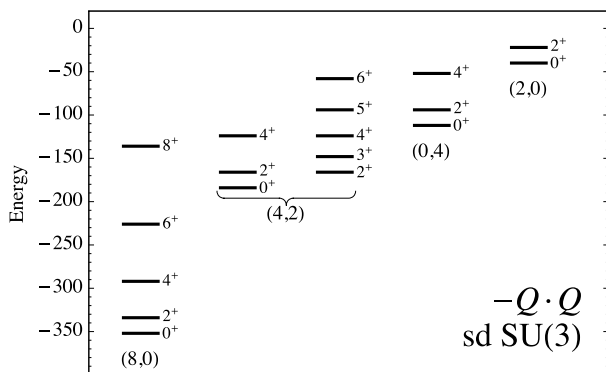


Fig. 3: The eigenspectrum of the operator  $-\hat{Q} \cdot \hat{Q}$  for two neutrons and two protons in the *sd* shell. Only levels in the favoured supermultiplet  $(0, 0, 0)$  are shown. Levels are labelled by the orbital angular momentum  $L$  and parity  $\pi = +$ , and by the SU(3) quantum numbers  $(\lambda, \mu)$ . All levels have  $S = 0$  and therefore the total angular momentum  $J$  equals the orbital angular momentum  $L$ .

$[\bar{h}] = [3, 1]$  or  $[\bar{h}'] = [2, 1, 1]$  or  $(\lambda', \mu', \nu') = (1, 0, 1)$ . And for four identical nucleons,  $T_z = \pm 2$ , it is  $[\bar{h}] = [\bar{h}'] = [2, 2]$  or  $(\lambda', \mu', \nu') = (0, 2, 0)$ . Retaining only the states contained in the favoured supermultiplets, we find the spectra shown in the right panel of figure 2 for  $T_z = 0, \pm 1$  and  $\pm 2$ . Finally, each irreducible representation  $(\lambda, \mu)$  corresponds to one or several rotational bands, as shown in figure 3 for the case of two neutrons and two protons ( $T_z = 0$ ) in the *sd* shell.

Elliott's SU(3) model contains the correct ingredients to describe quadrupole-deformed states in the context of the spherical shell model. The quadrupole interaction implies the orbital reduction (16) and represents an example of dynamical symmetry breaking. The degeneracy of states belonging to a Wigner supermultiplet is lifted dynamically by the quadrupole interaction. Elliott's SU(3) model gives rise to a rotational classification of states through mixing of spherical configurations and shows how

deformed nuclear shapes may emerge out of the spherical shell model. Elliott's work therefore establishes a link between the spherical nuclear shell model [4, 5] and the geometric collective model [19, 20, 21], which originally existed as separate views of the nucleus.

While Elliott's SU(3) model provides a natural explanation of nuclear rotations, it does so by assuming Wigner's SU(4) symmetry, which is known to be strongly broken in most but the lightest nuclei. This raises the following question: How can rotational phenomena in nuclei be understood starting from the *jj*-coupling scheme that applies to nearly all nuclei? Over the years several schemes have been proposed with the aim of transposing the SU(3) scheme to those modified situations. One such modification was suggested by Zuker *et al.* [48] under the name of quasi-SU(3), invoking the similarities of matrix elements of the quadrupole operator in the *jj*- and *LS*-coupling schemes. In a recent study [49] quasi-SU(3) was shown to provide a natural scheme in which to describe rotational motion when the dominant single-particle levels satisfy a  $\Delta j = 2$  condition and furthermore to give a framework in which to understand why most deformed nuclei have prolate shapes.

One of the most successful ways to extend the applications of the SU(3) scheme to heavier nuclei makes use of the concept of pseudo-spin symmetry, which can be explained by considering the single-particle part of the hamiltonian (1). For  $\zeta_{\ell\ell} = \zeta_{\ell s} = 0$  it displays the degeneracies associated with the U(3) symmetry of the three-dimensional isotropic harmonic oscillator. For general values  $\zeta_{\ell\ell} \neq 0$  and  $\zeta_{\ell s} \neq 0$  this U(3) symmetry is broken, except for the combination  $4\zeta_{\ell\ell} = \zeta_{\ell s}$  when some degree of degeneracy, associated with a pseudo-spin symmetry, is restored in the single-particle spectrum.

The existence of nearly degenerate pseudo-spin doublets in the nuclear mean-field potential was pointed out a long time ago by Hecht and Adler [50] and by Arima *et al.* [51]. These authors noted that, while *LS* coupling

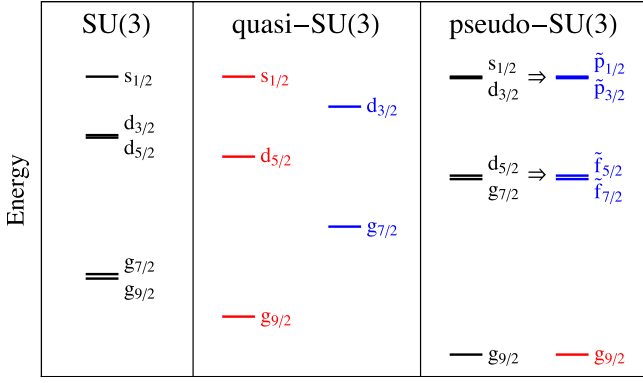


Fig. 4: The single-particle energies (for a non-zero orbit-orbit term,  $\zeta_{\ell\ell} \neq 0$ ) in SU(3), quasi-SU(3) and pseudo-SU(3) for the  $N = 4$  oscillator shell with the orbital shells  $sdg$ . The spin-orbit strength vanishes in SU(3),  $\zeta_{\ell s} = 0$ , in pseudo-SU(3)  $\zeta_{\ell s} = 4\zeta_{\ell\ell}$  and in quasi-SU(3) a possible choice is  $\zeta_{\ell s} \approx 2\zeta_{\ell\ell}$ . The single-particle spaces in red and in blue are assumed to be approximately decoupled. In pseudo-SU(3) the level degeneracies can be interpreted in terms of a pseudo-spin symmetry.

becomes unacceptable in medium-mass and heavy nuclei, pseudo- $LS$  coupling might be a reasonable starting point since the pseudo-spin-orbit splitting is small. With this assumption as a premise, a pseudo-SU(3) model can be constructed [52], similar to Elliott's SU(3) model in  $LS$  coupling. Many years later Ginocchio showed pseudo-spin to be a symmetry of the Dirac equation if the scalar and vector potentials are equal in size but opposite in sign [53].

Many applications of the pseudo-SU(3) scheme to heavy deformed nuclei have been reported in the literature. A summary of representative results was reported in Ref. [54], where results for bands in the even-even nucleus  $^{166}\text{Er}$  and in the odd-mass nucleus  $^{163}\text{Dy}$  were given and excellent agreement between theory and experiment was achieved in both cases.

Figure 4 provides a schematic illustration of the various SU(3)-like symmetries for the  $N = 4$  ( $sdg$ ) shell of the harmonic oscillator. On the left-hand side are shown the single-particle energies in the case of a vanishing spin-orbit term,  $\zeta_{\ell s} = 0$ , but a non-zero orbit-orbit term,  $\zeta_{\ell\ell} \neq 0$ . The pseudo-SU(3) symmetry applies if  $\zeta_{\ell s} = 4\zeta_{\ell\ell}$  while the quasi-SU(3) scenario is valid, for example, for  $\zeta_{\ell s} \approx 2\zeta_{\ell\ell}$ . At the basis of the quasi-SU(3) and pseudo-SU(3) symmetries is the assumption of an approximate decoupling of the single-particle spaces shown in red and blue in the figure. They can, however, be coupled by non-SU(3) interactions, *e.g.* the pairing interaction.

### 3.2. Two oscillator shells

Although not considered originally by Elliott, it is possible to formulate the SU(3) model for several oscillator shells. This extension is relatively straightforward if two consecutive shells of the harmonic oscillator are considered with the major quantum numbers  $N_- \equiv N - 1$  and  $N_+ \equiv N$ . For  $N = 1, 2, 3, \dots$  they contain the  $s$ - $p$ ,  $p$ - $sd$ ,  $sd$ - $pf$ , ... orbital shells, respectively. The orbital dimension of this system is  $\Omega^2$  with  $\Omega \equiv N + 1$ ; for neutrons and protons the total dimension is  $4\Omega^2$  to account for the spin-isospin degrees of freedom.

A classification of states can be proposed based on the separation of the orbital and spin-isospin degrees of freedom which is the analogue of equation (5),

$$\begin{array}{ccc} \text{U}(4\Omega^2) & \supset & \text{U}_L(\Omega^2) \otimes \text{U}_{ST}(4) \\ \downarrow & & \downarrow \quad \downarrow \\ [1^n] & & [\bar{h}] \quad [\bar{h}'] \end{array}, \quad (25)$$

where  $[\bar{h}] \equiv [h_1, \dots, h_{\Omega^2}]$  and  $[\bar{h}'] \equiv [h'_1, h'_2, h'_3, h'_4]$ . The generators of the orbital algebra  $\text{U}_L(\Omega^2)$  are the coupled tensors (7). In the one-shell case the orbital shells are  $\ell = N, N - 2, \dots, 1$  or 0 while in the two-shell analysis of this subsection they are  $\ell = N, N - 1, \dots, 0$ .

For this two-shell system the following orbital classification can be proposed:

$$\begin{array}{ccccccc} \text{U}(\Omega^2) & \supset & \text{U}(\Gamma_-) \otimes \text{U}(\Gamma_+) & \supset & \text{U}_-(3) \otimes \text{U}_+(3) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ [\bar{h}] & & [\bar{h}_-] & & [\bar{h}_+] & & [\bar{h}''_-] \quad [\bar{h}''_+] \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \text{SU}_-(3) \otimes \text{SU}_+(3) & \supset & \text{SU}(3) & \supset & \text{SO}(3) \\ & & \downarrow & & \downarrow & & \downarrow \\ & & (\lambda_-, \mu_-) & & (\lambda_+, \mu_+) & & (\lambda, \mu) \quad K \quad L \end{array}, \quad (26)$$

where the subscript ' $L$ ' is omitted from the orbital algebras for simplicity's sake and the notations  $\Gamma_-$  and  $\Gamma_+$  are introduced for the orbital dimensions of the lower and upper shells, respectively. Note that  $\Gamma_- + \Gamma_+ = \Omega^2$ . The orbital classification (26) implies a reduction from the total algebra  $\text{U}(\Omega^2)$  to the product algebra  $\text{U}(\Gamma_-) \otimes \text{U}(\Gamma_+)$ , where the algebras  $\text{U}(\Gamma_{\pm})$  are associated with the separate oscillator shells with major quantum numbers  $N_{\pm}$ . This requires the knowledge of the branching rule for  $\text{U}(\Omega^2) \supset \text{U}(\Gamma_-) \otimes \text{U}(\Gamma_+)$ , which is explained in the appendix. The determination of the  $\text{SU}_{\pm}(3)$  labels  $(\lambda_{\pm}, \mu_{\pm})$  proceeds as in section 3.1 while the labels  $(\lambda, \mu)$  associated with SU(3) follow from the standard multiplication of Young diagrams in U(3) [34].

The following hamiltonian can be proposed for the two-shell SU(3) model:

$$\varepsilon_- \hat{n}_- + \varepsilon_+ \hat{n}_+ + \kappa_- \hat{Q}_- \cdot \hat{Q}_- + \kappa_+ \hat{Q}_+ \cdot \hat{Q}_+ + \kappa \hat{Q}_- \cdot \hat{Q}_+ + \kappa' \hat{L} \cdot \hat{L},$$

where  $\hat{n}_{\pm}$ ,  $\hat{L}_{\pm, \mu}$  and  $\hat{Q}_{\pm, \mu}$  are the number, orbital angular momentum and quadrupole operators for the oscillator shells with major quantum numbers  $N_{\pm}$ , and  $\hat{L}_{\mu}$  and  $\hat{Q}_{\mu}$  are summed operators,

$$\hat{L}_{\mu} \equiv \hat{L}_{-, \mu} + \hat{L}_{+, \mu}, \quad \hat{Q}_{\mu} \equiv \hat{Q}_{-, \mu} + \hat{Q}_{+, \mu}.$$

This hamiltonian has the desired flexibility since it allows an energy difference  $\Delta\varepsilon \equiv \varepsilon_+ - \varepsilon_-$  between the lower and upper shell, possibly different strengths of the quadrupole interactions in the two shells and a rotational  $\hat{L} \cdot \hat{L}$  term. An essentially equivalent hamiltonian can be written in terms of Casimir operators,

$$\hat{H} = \varepsilon_- \hat{C}_1[\text{U}(\Gamma_-)] + \varepsilon_+ \hat{C}_1[\text{U}(\Gamma_+)] + \kappa_- \hat{C}_2[\text{SU}_-(3)] + \kappa_+ \hat{C}_2[\text{SU}_+(3)] + \kappa \hat{C}_2[\text{SU}(3)] + \kappa' \hat{C}_2[\text{SO}(3)]. \quad (27)$$

Since all Casimir operators are associated with the single chain (26) of nested algebras, this hamiltonian has a dynamical symmetry and is therefore analytically solvable. The operator  $\hat{C}_2[\text{U}(\Omega^2)]$  can be added to the hamiltonian (27) to achieve the separation of supermultiplets, in accordance with the discussion of section 2.

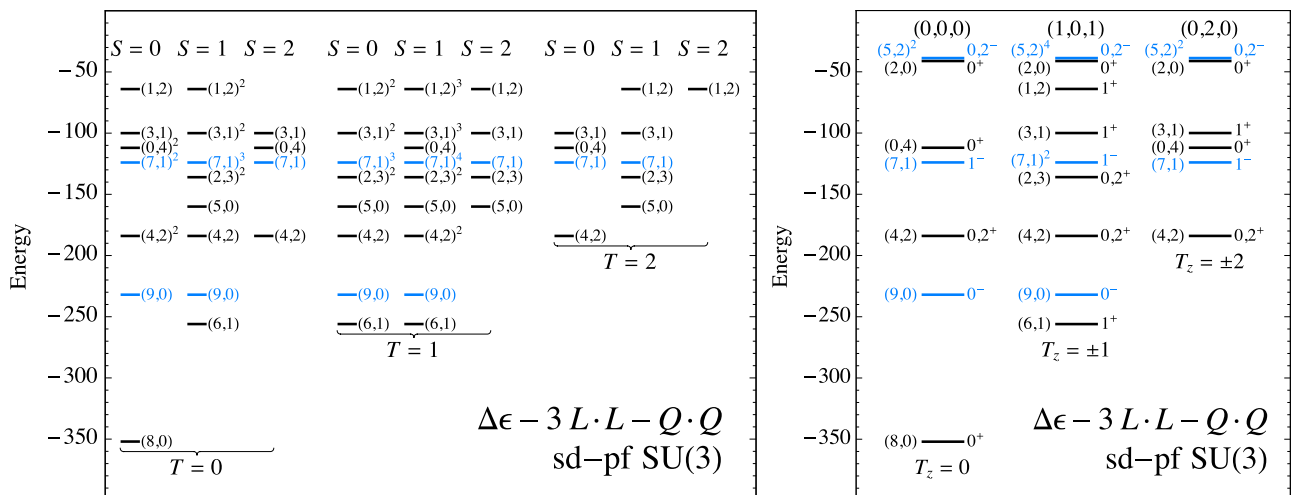


Fig. 5: Partial eigenspectrum of the hamiltonian  $\Delta\varepsilon(\hat{n}_+ - \hat{n}_-) + \kappa(3\hat{L} \cdot \hat{L} + \hat{Q} \cdot \hat{Q})$  for four nucleons in the  $sd$ - $pf$  shells with  $\Delta\varepsilon = -200\kappa$ . Levels of positive (negative) parity are black (blue). Left panel: Levels are labelled by the SU(3) quantum numbers  $(\lambda, \mu)$  and values of the total spin  $S$  and the total isospin  $T$  are also indicated. If an irreducible representation  $(\lambda, \mu)$  occurs  $k$  times for a given  $S$  and  $T$ , this is indicated by a superscript as  $(\lambda, \mu)^k$ . Right panel: Only levels in the favoured supermultiplets are shown, which are  $(0, 0, 0)$  for  $T_z = 0$ ,  $(1, 0, 1)$  for  $T_z = \pm 1$  and  $(0, 2, 0)$  for  $T_z = \pm 2$ . Levels are labelled by the SU(3) quantum numbers  $(\lambda, \mu)$  on the left and by the projections  $K$  and parity  $\pi$  on the right. The supermultiplet labels  $(\lambda', \mu', \nu')$  and the isospin projection  $T_z$  are also indicated, and all levels have  $S = 0$ .

An example eigenspectrum of the hamiltonian (27) is shown in figure 5 for four nucleons in the  $sd$ - $pf$  shells (*i.e.*,  $n = 4$ ,  $\Gamma_- = 6$  and  $\Gamma_+ = 10$ ). The spectrum contains the four-nucleon states of the  $sd$  shell (levels in black, identical to those in figure 2) and many additional states that correspond to excitations of nucleons from the  $sd$  to  $pf$  shell. For the choice of parameters in figure 5,  $\Delta\varepsilon = -200\kappa$ ,  $\kappa < 0$ , the latter excitations occur at higher energy and only those corresponding to excitations of one nucleon from the  $sd$  to  $pf$  shell are shown (levels in blue, of negative parity). The states shown in the left panel of figure 5 belong to the five possible supermultiplets labelled by  $[\bar{h}]$  or  $[\bar{h}']$  in equation (25). With the same assumption as in the single-shell case, retaining only the states contained in the favoured supermultiplets, we find the spectra shown in the right panel of figure 5 for  $T_z = 0, \pm 1$  and  $\pm 2$ . Some of the states have spurious components, as discussed in the example that follows.

Let us further illustrate the two-shell SU(3) model with an application to  $^{20}\text{Ne}$ . In its ground-state configuration this nucleus has four nucleons in the  $sd$  shell. Its observed negative-parity levels presumably result from excitations of nucleons from the  $p$  to  $sd$  shell and a reasonable *ansatz* therefore is to consider  $n = 16$  nucleons in the  $p$ - $sd$  shells, which implies  $N_- = 1$  and  $N_+ = 2$ , and  $U(\Omega^2) = U(9)$ ,  $U(\Gamma_-) = U(3)$  and  $U(\Gamma_+) = U(6)$ .

The size of the model space for 16 nucleons in the  $p$ - $sd$  shells is given by the dimension of the irreducible representation  $[1^{16}]$  of  $U(36)$ , which, including all magnetic substates in angular momentum and isospin, is 7 307 872 110. This dimension can be reduced by considering a specific magnetic substate (*e.g.*,  $M_J = M_T = 0$ ) but the resulting model space will still be huge. To simplify, we assume in the following that all low-energy states belong to the favoured supermultiplet  $[\bar{h}] = [4, 4, 4, 4]$  of  $U_L(9)$ , which corresponds to  $[\bar{h}'] = [4, 4, 4, 4]$  of  $U_{ST}(4)$  or to  $(\lambda', \mu', \nu') = (0, 0, 0)$  of  $SU_{ST}(4)$ . This implies that

$S = T = 0$  and therefore that the orbital and total angular momenta are equal,  $L = J$ .

The next step is the branching rule  $U(9) \supset U(3) \otimes U(6)$  for the irreducible representation  $[\bar{h}] = [4, 4, 4, 4]$  of  $U(9)$ . The latter has the dimension 1 646 568 and contains 35 product representations  $[\bar{h}_-] \times [\bar{h}_+]$  of  $U(3) \otimes U(6)$ . Most of them are of no importance for the low-energy spectrum of  $^{20}\text{Ne}$ . For positive  $\Delta\varepsilon$  the lowest-energy product representation is  $[\bar{h}_-] \times [\bar{h}_+] = [4, 4, 4] \times [4]$ , which corresponds to twelve nucleons in the  $p$  shell and four in the  $sd$  shell. The next product representation of  $U(3) \otimes U(6)$  contained in  $[\bar{h}] = [4, 4, 4, 4]$  of  $U(9)$  is  $[\bar{h}_-] \times [\bar{h}_+] = [4, 4, 3] \times [4, 1]$ , and corresponds to the excitation of one nucleon from the  $p$  to  $sd$  shell. This correspondence is unique, that is, a one-particle-one-hole excitation necessarily belongs to  $[4, 4, 3] \times [4, 1]$ . It is possible to continue in this fashion for excitations involving more nucleons but, in view of spurious centre-of-mass components (see below), this makes little sense and we limit the discussion here to cross-shell excitations of at most one nucleon.

The  $SU_-(3)$  labels  $(\lambda_-, \mu_-)$  of the lower ( $p$ ) shell are trivially obtained since they are the differences of the  $[\bar{h}_-]$  labels,  $\lambda_- = h_{-,1} - h_{-,2}$  and  $\mu_- = h_{-,2} - h_{-,3}$ . They are therefore  $(\lambda_-, \mu_-) = (0, 0)$  for twelve nucleons in the  $p$  shell and  $(\lambda_-, \mu_-) = (0, 1)$  for eleven nucleons (or one hole) in the  $p$  shell. The  $SU_+(3)$  labels  $(\lambda_+, \mu_+)$  for four and five nucleons, respectively, in the upper ( $sd$ ) shell follow from the branching rules for  $U(6) \supset SU(3)$ , which can be taken from Elliott [45] (see also the appendix),

$$\begin{aligned}
[4] &\mapsto (8, 0) + (4, 2) + (0, 4) + (2, 0), \\
[4, 1] &\mapsto (8, 1) + (6, 2) + (4, 3) + (5, 1) + (2, 4) + (3, 2) \\
&\quad + (4, 0) + (1, 3) + (2, 1) + (0, 2).
\end{aligned}$$

The final step requires the outer multiplication

$(\lambda_-, \mu_-) \times (\lambda_+, \mu_+)$ , which yields the SU(3) labels  $(\lambda, \mu)$ ,

$$\begin{aligned} (0, 0) \times [(8, 0) + (4, 2) + \dots] &= (8, 0) + (4, 2) + \dots, \\ (0, 1) \times [(8, 1) + (6, 2) + \dots] &= (8, 2) + (9, 0) \\ &\quad + (6, 3) + (7, 1) + \dots, \end{aligned}$$

where only the most important SU(3) irreducible representations of each series are listed. This completes the discussion of all labels necessary to determine the low-energy eigenspectrum of the hamiltonian (27) applied to  $^{20}\text{Ne}$ .

In figure 6 the eigenspectrum of the hamiltonian (27), with parameters (in MeV)  $\Delta\varepsilon = 7.60$ ,  $\kappa_- = \kappa_+ = -0.05$ ,  $\kappa = -0.10$  and  $\kappa' = -0.16$ , is compared with the observed spectrum of  $^{20}\text{Ne}$ , taken from NNDC [55]. From the levels shown in the figure it is not possible to determine separately the parameters  $\Delta\varepsilon$  and  $\kappa_-$ , and therefore  $\kappa_- = \kappa_+$  is taken, which corresponds to equal quadrupole strengths in the  $p$  and  $sd$  shells. The assumption of different strengths,  $\kappa_- \neq \kappa_+$ , has no impact on the quality of the fit and leads to a (slightly) different value of  $\Delta\varepsilon$ . All observed levels up to an energy of 10 MeV are shown in figure 6 and most can be associated with a theoretical counterpart. Among the additional observed levels (shown in the third column of the left panel) several could be members of the collective  $\gamma$ -vibrational  $K^\pi = 2^+$  band or be interpreted as cross-shell excitations of several nucleons. On the theoretical side, a  $K^\pi = 2^+$  band is calculated in the  $(\lambda, \mu) = (4, 2)$  irreducible representation of SU(3), which, surprisingly, has not been established experimentally. More conspicuous is the presence in the theory of an additional  $K^\pi = 0^-$  band in the  $(\lambda, \mu) = (8, 2)$  irreducible representation of SU(3), which is not seen experimentally.

If more than one oscillator shell is considered in the model space, care must be taken to eliminate spurious centre-of-mass components from the calculated states. If the model space consists of entire shells of the harmonic oscillator, an exact procedure exists for doing so, based on the action of the centre-of-mass operator  $\bar{R}$  on the ground state of a given nucleus, and the elimination from the excitation spectrum of the states so created [56]. The method is particularly attractive in the SU(3) scheme since the operator  $\bar{R}$  has a  $[1, 0, 0]$  tensor character under U(3). Therefore, if the ground state belongs to a certain irreducible representation  $(\lambda, \mu)$  of SU(3), spurious states follow from the repeated multiplication of  $(\lambda, \mu)$  with  $(1, 0)$ . In the example of  $^{20}\text{Ne}$  the single action of  $\bar{R}$  on the ground state leads to spurious excitations with SU(3) character  $(9, 0)$  and  $(7, 1)$  since

$$(8, 0) \times (1, 0) = (9, 0) + (7, 1).$$

As one of the irreducible representations,  $(9, 0)$ , occurs in the low-energy spectrum, we must prove that it is not spurious.

It should be noted that the technique for eliminating spurious centre-of-mass components cannot be accommodated in the classification (26) because the action of the operator  $\bar{R}$  creates additional excitations outside the model space. In the example of  $^{20}\text{Ne}$ , which has a ground-state configuration with a completely filled  $p$  shell and four nucleons in the  $sd$  shell,  $\bar{R}$  excites nucleons not only from

the  $p$  to  $sd$  shell [excitations that are included in the classification (26)] but also from the  $sd$  to  $pf$  shell (excitations that are not). For a correct elimination of spurious centre-of-mass components from all one-particle-one-hole excitations, it is therefore necessary to consider the  $p$ ,  $sd$  and  $pf$  shells—a straightforward extension of the two-shell system considered so far. All previous results remain valid and, in addition, the spectrum should be complemented with excitations of one nucleon from the  $sd$  to  $pf$  shell. The lowest-energy excitations of the latter type have SU(3) character

$$(6, 0) \times (3, 0) = (9, 0) + (7, 1) + \dots,$$

corresponding to three nucleons in the  $sd$  and one nucleon in the  $pf$  shell. We conclude that in the three-shell  $p$ - $sd$ - $pf$  system *two* irreducible representations  $(9, 0)$  occur among the low-energy one-particle-one-hole excitations. One is spurious and must be eliminated; the other is physical and is included in figure 6.

These results are obtained under the assumption that all observed states in  $^{20}\text{Ne}$  belong to the favoured supermultiplet  $[\bar{h}] = [4, 4, 4, 4]$  of  $U_L(9)$ . While the hypothesis of maximal spatial symmetry might be acceptable for the single-shell SU(3) model, it is more questionable in the case of several shells. In Ref. [57] it is argued that this hypothesis is related to quartet clustering and that the observed levels in  $^{20}\text{Ne}$  should therefore be interpreted as cluster states. The combined classifications (25) and (26), suitably extended to the appropriate multi-shell scenario, enable one to study the interplay between the maximization of spatial symmetry on the one hand and cross-shell excitations on the other, in order to verify the validity of the cluster interpretation.

### 3.3. Many oscillator shells

It is instructive to abandon for a moment the formalism of second quantization and return to the representation of operators in coordinate and momentum space. As mentioned already, U(3) generators can be represented in a spherical basis and the components of the quadrupole operator in this basis are [45]

$$\hat{Q}_\mu = \sqrt{\frac{3}{2}} \left( \sum_{k=1}^A (\bar{r}_k \times \bar{r}_k)_\mu^{(2)} / b^2 + b^2 \sum_{k=1}^A (\bar{p}_k \times \bar{p}_k)_\mu^{(2)} / \hbar^2 \right),$$

where  $b$  is the length parameter of the harmonic oscillator,  $b = \sqrt{\hbar/m_n\omega}$ . Because of its dependence on the position vectors  $\bar{r}_k$  and the momentum vectors  $\bar{p}_k$  of the nucleons, matrix elements of  $\hat{Q}_\mu$  between oscillator shells vanish, and it is precisely this property which gives rise to the algebraic structure of the SU(3) model. The components of the physical quadrupole operator, on the other hand, are

$$\hat{Q}'_\mu = \sqrt{6} \sum_{k=1}^A (\bar{r}_k \times \bar{r}_k)_\mu^{(2)} / b^2.$$

In a single oscillator shell, it makes no difference whether the algebraic quadrupole operator  $\hat{Q}_\mu$  or the physical quadrupole operator  $\hat{Q}'_\mu$  is used (matrix elements of both operators are identical due to the virial theorem). Because of parity, this statement remains valid for two shells

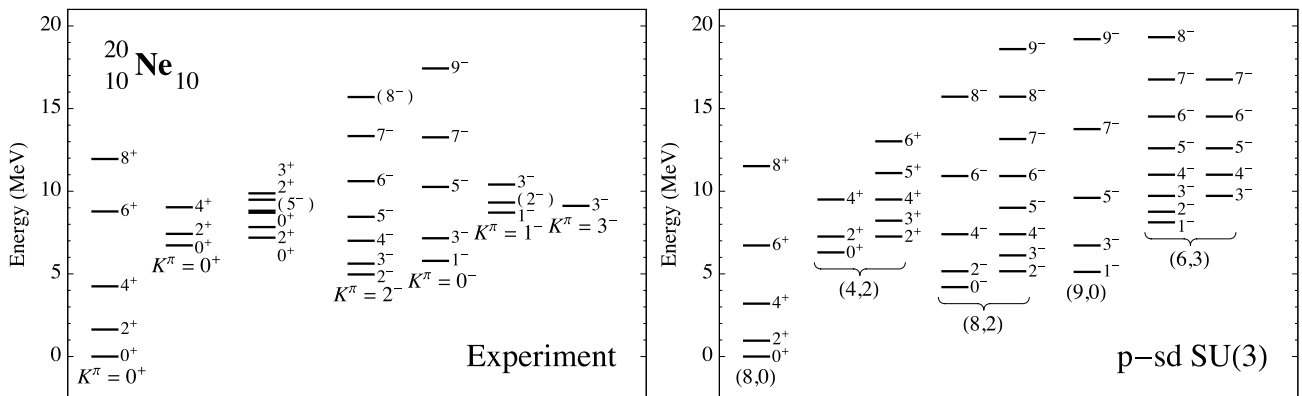


Fig. 6: Observed energy spectrum of  $^{20}\text{Ne}$  (taken from NNDC [55]) compared with the eigenspectrum of a two-shell SU(3) hamiltonian. Left panel: All observed levels up to an energy of 10 MeV are shown. Levels that can be associated with a theoretical counterpart are arranged as on the right panel and additional levels are shown in the third column. Uncertain spin-parity assignments are given in brackets and bands are tentatively assigned a  $K$  quantum number, in accordance with their E2 decay properties. Right panel: The two-shell SU(3) hamiltonian (27) is used for 16 nucleons in the  $p$ - $sd$  shells with parameters (in MeV)  $\Delta\varepsilon \equiv \varepsilon_+ - \varepsilon_- = 7.60$ ,  $\kappa_- = \kappa_+ = -0.025$ ,  $\kappa = -0.05$  and  $\kappa' = -0.16$ . Entire rotational bands are shown and labelled by the SU(3) quantum numbers  $(\lambda, \mu)$ .

$N - 1$  and  $N$ . Differences arise however, if more than two shells are considered because  $\hat{Q}'_\mu$  has non-zero  $\Delta N = \pm 2$  matrix elements. These are important because they are responsible for the full quadrupole collectivity in nuclei. For the sake of constructing a closed algebra, cross-shell correlations are thus lost from the SU(3) model.

By embedding SU(3) into the symplectic algebra  $\text{Sp}(6, R)$ , it is possible to accommodate cross-shell effects [58, 59]. (We follow Gilmore's notation of the Lie algebras of the classical groups [60].) The following relation between the algebraic and physical quadrupole operators illustrates the essence of this idea [61]:

$$\hat{Q}'_\mu = \hat{Q}_\mu + \sqrt{\frac{3}{2}}(\hat{B}_{+, \mu}^2 + \hat{B}_{-, \mu}^2),$$

where  $\hat{B}_{\pm, \mu}^2$  are  $2\hbar\omega$  raising and lowering operators with tensor character  $\ell = 2$ . The  $\text{Sp}(6, R)$  algebra consists of  $\{\hat{n}, \hat{L}_\mu, \hat{Q}_\mu, \hat{B}_{\pm, 0}^0, \hat{B}_{\pm, \mu}^2\}$ , where  $\hat{n}$  is the number operator,  $\hat{L}_\mu$  and  $\hat{Q}_\mu$  are the components of the angular momentum and quadrupole operators forming SU(3), and  $\hat{B}_{\pm, \mu}^\ell$  are the monopole ( $\ell = 0$ ) and quadrupole ( $\ell = 2$ )  $2\hbar\omega$  raising and lowering operators. These operators close under commutation and a linear combination of them corresponds to the physical quadrupole operator  $\hat{Q}'_\mu$ . The  $\text{Sp}(6, R)$  model contains all the necessary ingredients for the description of quadrupole-deformed states in the context of the spherical shell model with enhanced collectivity due to cross-shell excitations.

#### 4. A shell-model classification for octupole deformation

The question arises whether a generic classification exists for octupole-deformed shell-model states, similar to the one constructed by Elliott for quadrupole deformation. The minimal realization in this case should be in terms of the three components  $\hat{L}_\mu$  of the angular momentum operator and the seven components  $\hat{O}_\mu$  of the octupole operator. As the octupole operators should be of negative

parity, at least two harmonic-oscillator shells are required for this minimal realization.

We show that a symmetry of this type indeed exists, by following the treatment of section 3.2 and considering two consecutive oscillator shells with the major quantum numbers  $N_- \equiv N - 1$  and  $N_+ \equiv N$ .

The octupole classification discussed in this section is inspired by studies in the context of the interacting boson model [62], in particular the U(16) *spdf* version of it, which deals with the quantization of asymmetric shapes in nuclei [63]. This model, which has been extensively studied by Kusnezov [64, 65], has also been extended to odd-mass nuclei [66]. Our approach is different, however, since it is entirely fermionic and takes fully into account the Pauli principle. Just as Elliott's SU(3) symmetry is not to be confused with the SU(3) limit of the interacting boson model, the classification proposed here differs from the limits of the U(16) model. Also, we propose a symmetry treatment, which, like Elliott's, is generic and applies to any two consecutive major oscillator shells.

##### 4.1. A solvable dipole–octupole hamiltonian

The starting point of the analysis is based on the separation of the orbital and spin–isospin degrees of freedom, identical to equation (25). The generators of the orbital algebra  $U_L(\Omega^2)$  can be written as coupled tensors (7) but also as *double* tensors

$$\hat{G}_{\mu_1 \mu_2}^{(\lambda_1 \lambda_2)} \equiv \sum_{\ell_1 \ell_2} \sum_{\lambda \mu} (\lambda_1 \mu_1 \lambda_2 \mu_2 | \lambda \mu) \times \begin{bmatrix} \frac{1}{2}N & \frac{1}{2}N & \ell_1 \\ \frac{1}{2}N & \frac{1}{2}N & \ell_2 \\ \lambda_1 & \lambda_2 & \lambda \end{bmatrix} \hat{G}_\mu^{(\lambda)}(\ell_1 \ell_2). \quad (28)$$

The orbital shells in the two-shell analysis of this section are  $\ell = N, N - 1, \dots, 0$ , as shall be implicitly assumed henceforth in all summations.

It is not *a priori* clear what the physical meaning of the double tensors is. Nevertheless, as will be shown below, a linear combination of the operators (28) with

$(\lambda_1, \lambda_2) = (3, 0)$  and  $(0, 3)$  approximately corresponds to the octupole operator  $\hat{O}_\mu$ . Also, it is the double-tensor character of the generators that enables the definition of a product of unitary algebras in the classification proposed below, leading to the occurrence of parity doublets in the spectrum.

For nucleons occupying two major shells  $N - 1$  and  $N$ , the following orbital classification can be proposed:

$$\begin{array}{ccccccccc} U(\Omega^2) & \supset & U_a(\Omega) & \otimes & U_b(\Omega) & \supset & \text{SO}_{\text{p}_a}(\Omega) & \otimes & \text{SO}_{\text{p}_b}(\Omega) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ [\hat{h}] & & [\hat{h}_a] & & [\hat{h}_b] & & \langle \hat{\omega}_a \rangle & & \langle \hat{\omega}_b \rangle \\ \supset & \text{SO}_{\text{p}_+}(\Omega) & \supset & \text{SO}_+(3) & & & & & \\ \downarrow & & \downarrow & & & & & & \\ \langle \hat{\omega}_+ \rangle & & L & & & & & & \end{array}, \quad (29)$$

with  $\Omega \equiv N + 1$  and where the subscript ‘ $L$ ’ again is omitted from the orbital algebras for simplicity’s sake. The (non-standard) abbreviation  $\text{SO}_{\text{p}}(\Omega)$  refers to the (unitary) symplectic algebra  $\text{Sp}(\Omega)$  if  $\Omega$  is even and to the orthogonal algebra  $\text{SO}(\Omega)$  if  $\Omega$  is odd. The labels underneath the algebras are explained below.

The classification (29) follows from the commutator property of the double tensors  $\hat{G}_{\mu_1\mu_2}^{(\lambda_1\lambda_2)}$ , which reads

$$\begin{aligned} [\hat{G}_{\mu_1\mu_2}^{(\lambda_1\lambda_2)}, \hat{G}_{\mu_3\mu_4}^{(\lambda_3\lambda_4)}] &= \frac{1}{2} \hat{\lambda}_1 \hat{\lambda}_2 \hat{\lambda}_3 \hat{\lambda}_4 \sum_{\lambda_{13}\mu_{13}} \sum_{\lambda_{24}\mu_{24}} \\ &\times [(-)^{\lambda_{13}+\lambda_{24}} - (-)^{\lambda_1+\lambda_2+\lambda_3+\lambda_4}] \\ &\times (\lambda_1\mu_1 \lambda_3\mu_3 | \lambda_{13}\mu_{13}) (\lambda_2\mu_2 \lambda_4\mu_4 | \lambda_{24}\mu_{24}) \\ &\times \left\{ \begin{array}{ccc} \lambda_1 & \lambda_3 & \lambda_{13} \\ \frac{1}{2}N & \frac{1}{2}N & \frac{1}{2}N \end{array} \right\} \left\{ \begin{array}{ccc} \lambda_2 & \lambda_4 & \lambda_{24} \\ \frac{1}{2}N & \frac{1}{2}N & \frac{1}{2}N \end{array} \right\} \hat{G}_{\mu_{13}\mu_{24}}^{(\lambda_{13}\lambda_{24})}. \end{aligned} \quad (30)$$

The derivation of this commutator relation requires the expansion of the double tensors (28) in terms of uncoupled generators. It makes use of the anti-commutators (6) and of summation properties of Clebsch–Gordan coefficients, six- $j$  and nine- $j$  symbols [28]. The corresponding relation for bosons (without the coefficient  $1/2$ ) is given by Kuznezov for  $N = 3$  in connection with the *spdf* interacting boson model [67]. Many properties concerning the various octupole classifications can be derived from equation (30).

From the commutator property (30) it immediately follows that  $\hat{G}_{0\mu_1}^{(0\lambda_1)}$  and  $\hat{G}_{\mu_2 0}^{(\lambda_2 0)}$  commute,

$$[\hat{G}_{0\mu_1}^{(0\lambda_1)}, \hat{G}_{\mu_2 0}^{(\lambda_2 0)}] = 0.$$

It also follows that the operators  $\hat{G}_{0\mu}^{(0\lambda)}$  and  $\hat{G}_{\mu 0}^{(\lambda 0)}$  separately close under commutation since

$$\begin{aligned} [\hat{G}_{0\mu_1}^{(0\lambda_1)}, \hat{G}_{0\mu_2}^{(0\lambda_2)}] &= \frac{1}{2} \hat{\lambda}_1 \hat{\lambda}_2 \sum_{\lambda\mu} [(-)^\lambda - (-)^{\lambda_1+\lambda_2}] \\ &\times (\lambda_1\mu_1 \lambda_2\mu_2 | \lambda\mu) \left\{ \begin{array}{ccc} \lambda_1 & \lambda_2 & \lambda \\ \frac{1}{2}N & \frac{1}{2}N & \frac{1}{2}N \end{array} \right\} \hat{G}_{0\mu}^{(0\lambda)}, \end{aligned} \quad (31)$$

and

$$\begin{aligned} [\hat{G}_{\mu_1 0}^{(\lambda_1 0)}, \hat{G}_{\mu_2 0}^{(\lambda_2 0)}] &= \frac{1}{2} \hat{\lambda}_1 \hat{\lambda}_2 \sum_{\lambda\mu} [(-)^\lambda - (-)^{\lambda_1+\lambda_2}] \\ &\times (\lambda_1\mu_1 \lambda_2\mu_2 | \lambda\mu) \left\{ \begin{array}{ccc} \lambda_1 & \lambda_2 & \lambda \\ \frac{1}{2}N & \frac{1}{2}N & \frac{1}{2}N \end{array} \right\} \hat{G}_{\mu 0}^{(\lambda 0)}. \end{aligned} \quad (32)$$

This shows that the decomposition from  $U(\Omega^2)$  into the product algebra  $U_a(\Omega) \otimes U_b(\Omega)$  is achieved by requiring a scalar character in one of the indices of the double tensor (28),

$$\begin{aligned} U_a(\Omega) &= \{\hat{G}_{0\mu}^{(0\lambda)}, \lambda = 0, 1, \dots, N\}, \\ U_b(\Omega) &= \{\hat{G}_{\mu 0}^{(\lambda 0)}, \lambda = 0, 1, \dots, N\}. \end{aligned}$$

The explicit expression of the generators of  $U_a(\Omega)$  and  $U_b(\Omega)$  in terms of the tensors (7) is

$$\begin{aligned} \hat{G}_{0\mu}^{(0\lambda)} &= \sum_{\ell\ell'} \frac{(-)^{N+\lambda+\ell}}{\sqrt{N+1}} \hat{\ell}\hat{\ell}' \left\{ \begin{array}{ccc} \ell & \lambda & \ell' \\ \frac{1}{2}N & \frac{1}{2}N & \frac{1}{2}N \end{array} \right\} \hat{G}_\mu^{(\lambda)}(\ell\ell'), \\ \hat{G}_{\mu 0}^{(\lambda 0)} &= \sum_{\ell\ell'} \frac{(-)^{N+\lambda+\ell'}}{\sqrt{N+1}} \hat{\ell}\hat{\ell}' \left\{ \begin{array}{ccc} \ell & \lambda & \ell' \\ \frac{1}{2}N & \frac{1}{2}N & \frac{1}{2}N \end{array} \right\} \hat{G}_\mu^{(\lambda)}(\ell\ell'). \end{aligned} \quad (33)$$

The subsequent reduction to  $\text{Sp}(\Omega)$  or  $\text{SO}(\Omega)$  in equation (29) follows from the restriction to odd-integer  $\lambda$ ,

$$\begin{aligned} \text{SO}_{\text{p}_a}(\Omega) &= \{\hat{G}_{0\mu}^{(0\lambda)}, \lambda = \text{odd}\}, \\ \text{SO}_{\text{p}_b}(\Omega) &= \{\hat{G}_{\mu 0}^{(\lambda 0)}, \lambda = \text{odd}\}, \end{aligned}$$

which close under commutation because of the presence of the phase factor in equations (31) and (32). The algebra  $\text{SO}_{\text{p}_+}(\Omega)$  is obtained by adding the generators of  $\text{SO}_{\text{p}_a}(\Omega)$  and  $\text{SO}_{\text{p}_b}(\Omega)$ ,

$$\text{SO}_{\text{p}_+}(\Omega) = \{\hat{G}_{0\mu}^{(0\lambda)} + \hat{G}_{\mu 0}^{(\lambda 0)}, \lambda = \text{odd}\}.$$

The components of the operator

$$\begin{aligned} \hat{L}_\mu &\equiv \left[ \frac{N(N+1)^2(N+2)}{3} \right]^{1/2} (\hat{G}_{0\mu}^{(01)} + \hat{G}_{\mu 0}^{(10)}) \\ &= \sum_\ell \sqrt{\frac{4\ell(\ell+1)(2\ell+1)}{3}} \hat{G}_\mu^{(1)}(\ell\ell), \end{aligned} \quad (34)$$

are the orbital angular momentum generators and therefore the algebra  $\text{SO}_+(3)$  coincides with the orbital angular momentum algebra.

The algebra  $\text{SO}_{\text{p}_+}(\Omega)$  contains, besides  $\hat{L}_\mu$ , the components of the octupole operator  $\hat{G}_{0\mu}^{(03)} + \hat{G}_{\mu 0}^{(30)}$ ; it is of positive parity, however, and therefore does not correspond to the  $r^3 Y_{3\mu}$  operator of relevance for octupole deformation in nuclei. To obtain negative-parity operators, we consider the combinations

$$\text{SO}_{\text{p}_-}(\Omega) = \{\hat{G}_{0\mu}^{(0\lambda)} - \hat{G}_{\mu 0}^{(\lambda 0)}, \lambda = \text{odd}\}.$$

We use the notation

$$\hat{T}_\mu^{(\lambda)} \equiv \sqrt{8(N+1)} (\hat{G}_{0\mu}^{(0\lambda)} - \hat{G}_{\mu 0}^{(\lambda 0)}), \quad (35)$$

where the coefficient is introduced for later convenience, and we adopt furthermore the notations  $\hat{D}_\mu \equiv \hat{T}_\mu^{(1)}$  and  $\hat{O}_\mu \equiv \hat{T}_\mu^{(3)}$  for the dipole and octupole operators, respectively.

The generators of  $\text{SO}_{\text{p}_a}(\Omega)$  commute with those of  $\text{SO}_{\text{p}_b}(\Omega)$  but this is not the case for those of  $\text{SO}_{\text{p}_+}(\Omega)$  and  $\text{SO}_{\text{p}_-}(\Omega)$ . Also, the generators of  $\text{SO}_{\text{p}_\pm}(\Omega)$  are parity-conserving while this is not so for  $\text{SO}_{\text{p}_{a,b}}(\Omega)$ .



In the orbital classification (29) appear the  $U(\Omega^2)$  labels  $[\bar{h}]$  that are carried over from Wigner's supermultiplet model. The algebras  $U_a(\Omega)$  and  $U_b(\Omega)$  are characterized by the Young diagrams  $[\bar{h}_a]$  and  $[\bar{h}_b]$ , respectively, which can have at most  $\Omega$  rows. They follow from the branching rule for  $U(\Omega^2) \supset U(\Omega) \otimes U(\Omega)$ , the algorithm of which is explained in the appendix. The next step involves twice (*i.e.*, for  $a$  and for  $b$ ) the branching rule for  $U(\Omega) \supset \text{SO}p(\Omega)$ , that is,  $U(\Omega) \supset \text{Sp}(\Omega)$  if  $\Omega$  is even, and  $U(\Omega) \supset \text{SO}(\Omega)$  if  $\Omega$  is odd. The number of labels for  $\text{Sp}(\Omega)$  is  $\Omega/2$  while it is  $(\Omega - 1)/2$  for  $\text{SO}(\Omega)$  [31]. The labels are denoted in equation (29) as  $\langle \bar{\omega} \rangle \equiv \langle \omega_1, \omega_2, \dots \rangle$ , notation which is thus used for irreducible representations of either symplectic or orthogonal algebras. In the following, the notation  $\{\bar{\nu}\} \equiv \{\nu_1, \nu_2, \dots\}$  shall be reserved for symplectic algebras ( $N$  odd) and  $(\bar{\nu}) \equiv (\nu_1, \nu_2, \dots)$  for orthogonal algebras ( $N$  even). The branching rules for  $U(\Omega) \supset \text{Sp}(\Omega)$  and  $U(\Omega) \supset \text{SO}(\Omega)$  can be obtained for any  $\Omega$  from  $S$ -function theory for restricted algebras and they determine the allowed labels  $\langle \bar{\omega}_a \rangle$  and  $\langle \bar{\omega}_b \rangle$  in the classification (29). The next step requires the multiplication  $\langle \bar{\omega}_a \rangle \times \langle \bar{\omega}_b \rangle$ , that is, the outer multiplication of irreducible representations in either  $\text{Sp}(\Omega)$  or  $\text{SO}(\Omega)$ . This is also known for any  $\Omega$  from  $S$ -function theory for restricted algebras and yields the irreducible representations  $\langle \bar{\omega}_+ \rangle$  of  $\text{SO}p_+(\Omega)$ . Finally, the branching rules for  $\text{Sp}(\Omega) \supset \text{SO}(3)$  or  $\text{SO}(\Omega) \supset \text{SO}(3)$  determine the allowed orbital angular momenta  $L$  and can be obtained from the plethysm of  $S$  functions.

Group-theoretical methods related to  $S$ -function theory are described, for example, in the book by Wybourne [47] which also includes tables (by Butler) with outer multiplications of  $S$  functions, expansions of characters of restricted (*i.e.*, symplectic or orthogonal) algebras into  $S$  functions and various branching rules. Many (but not all) cases of interest for the classification (29) can be obtained from the tables. The techniques needed for the present application are described in the appendix.

For low values of  $N$ , isomorphisms between orthogonal and symplectic algebras exist,  $\text{Sp}(2) \sim \text{SO}(3)$  and  $\text{Sp}(4) \sim \text{SO}(5)$ . Therefore, up to  $N = 4$  we may choose to use the orthogonal algebras  $\text{SO}(3)$  ( $N = 1$  or  $2$ ) and  $\text{SO}(5)$  ( $N = 3$  or  $4$ ). For odd values of  $N$ , spinor representations of the orthogonal algebras should be employed, the correspondence with the symplectic labels being

$$\begin{aligned} \text{Sp}(2) \sim \text{SO}(3) : \quad \nu_1 &= \frac{\nu_1}{2}, \\ \text{Sp}(4) \sim \text{SO}(5) : \quad \nu_1 &= \frac{\nu_1 + \nu_2}{2}, \quad \nu_2 = \frac{\nu_1 - \nu_2}{2}. \end{aligned}$$

The quadratic Casimir operator of any of the  $\text{SO}p(\Omega)$  algebras is defined as

$$\hat{C}_2[\text{SO}p(\Omega)] = 8(N+1) \sum_{\lambda \text{ odd}} \hat{G}^{(\lambda)} \cdot \hat{G}^{(\lambda)}, \quad (36)$$

where  $\hat{G}_\mu^{(\lambda)}$  can be  $\hat{G}_{0\mu}^{(0\lambda)}$  or  $\hat{G}_{\mu 0}^{(\lambda 0)}$ , or the combinations  $\hat{G}_{0\mu}^{(0\lambda)} \pm \hat{G}_{\mu 0}^{(\lambda 0)}$ . The eigenvalue expressions are known from classical group theory (see, for example, table 5.1 of Ref. [31]). The eigenvalues of the operator  $\hat{C}_2[\text{Sp}(\Omega)]$ , for even  $\Omega$ , are

$$\sum_{i=1}^{\Omega/2} \nu_i(\nu_i + \Omega + 2 - 2i),$$

while those of the operator  $\hat{C}_2[\text{SO}(\Omega)]$ , for odd  $\Omega$ , are

$$\sum_{i=1}^{(\Omega-1)/2} \nu_i(\nu_i + \Omega - 2i).$$

The two expressions can be combined by introducing

$$E_N(\bar{\omega}) \equiv \sum_{i=1}^{\lfloor \Omega/2 \rfloor} \omega_i(\omega_i + \Omega_N + 2 - 2i), \quad (37)$$

with  $\Omega_N \equiv \Omega = N + 1$  for odd  $N$  (symplectic algebras) and  $\Omega_N \equiv \Omega - 2 = N - 1$  for even  $N$  (orthogonal algebras). Casimir operators are only determined up to a proportionality factor and the coefficient  $8(N + 1)$  in equation (36) is chosen such that the expectation value of  $\hat{C}_2[\text{SO}p(\Omega)]$  yields the eigenvalue (37).

A definition similar to equation (36) holds for the quadratic Casimir operator of the  $\text{SO}(3)$  algebras,

$$\hat{C}_2[\text{SO}_\pm(3)] = \frac{N(N+1)^2(N+2)}{3} \hat{G}^{(1)} \cdot \hat{G}^{(1)}, \quad (38)$$

where  $\hat{G}_\mu^{(1)} = \hat{G}_{0\mu}^{(01)} \pm \hat{G}_{\mu 0}^{(10)}$ . The coefficient in equation (38) is chosen such that the expectation value of  $\hat{C}_2[\text{SO}_+(3)]$  yields the eigenvalue  $L(L + 1)$ .

From the previous results it follows that

$$\begin{aligned} - \sum_{\lambda \text{ odd}} \hat{T}^{(\lambda)} \cdot \hat{T}^{(\lambda)} &= -2\hat{C}_2[\text{SO}p_a(\Omega)] - 2\hat{C}_2[\text{SO}p_b(\Omega)] \\ &\quad + \hat{C}_2[\text{SO}p_+(\Omega)]. \end{aligned} \quad (39)$$

Since this hamiltonian can be written as a combination of Casimir operators belonging to the chain (29) of nested algebras, it is solvable with eigenstates

$$|[1^n]; ([\bar{h}]; [\bar{h}_a]\langle \bar{\omega}_a \rangle \times [\bar{h}_b]\langle \bar{\omega}_b \rangle; \langle \bar{\omega}_+ \rangle L) \times [\bar{h}']ST], \quad (40)$$

and energy eigenvalues

$$-2E_N(\bar{\omega}_a) - 2E_N(\bar{\omega}_b) + E_N(\bar{\omega}_+). \quad (41)$$

This establishes the result that a hamiltonian which is a sum over odd, negative-parity tensors is solvable.

## 4.2. Parity doublets

The basis states (40) in general do not carry a definite parity quantum number. From the elementary relation for one particle,

$$\hat{P}^{-1} a_{\ell m_\ell s m_s t m_t}^\dagger \hat{P} = (-)^\ell a_{\ell m_\ell s m_s t m_t}^\dagger,$$

where  $\hat{P}$  is the parity operator, it follows that

$$\hat{P}^{-1} \hat{G}_\mu^{(\lambda)}(\ell \ell') \hat{P} = (-)^{\ell + \ell'} \hat{G}_\mu^{(\lambda)}(\ell \ell'),$$

and therefore

$$\hat{P}^{-1} \hat{G}_{0\mu}^{(0\lambda)} \hat{P} = \hat{G}_{\mu 0}^{(\lambda 0)}, \quad \hat{P}^{-1} \hat{G}_{\mu 0}^{(\lambda 0)} \hat{P} = \hat{G}_{0\mu}^{(0\lambda)}.$$

This implies that the basis states (40) are transformed as

$$\begin{aligned} \hat{P} |[1^n]; ([\bar{h}]; [\bar{h}_a]\langle \bar{\omega}_a \rangle \times [\bar{h}_b]\langle \bar{\omega}_b \rangle; \langle \bar{\omega}_+ \rangle L) \times [\bar{h}']ST] \\ = \varphi |[1^n]; ([\bar{h}]; [\bar{h}_b]\langle \bar{\omega}_b \rangle \times [\bar{h}_a]\langle \bar{\omega}_a \rangle; \langle \bar{\omega}_+ \rangle L) \times [\bar{h}']ST], \end{aligned}$$

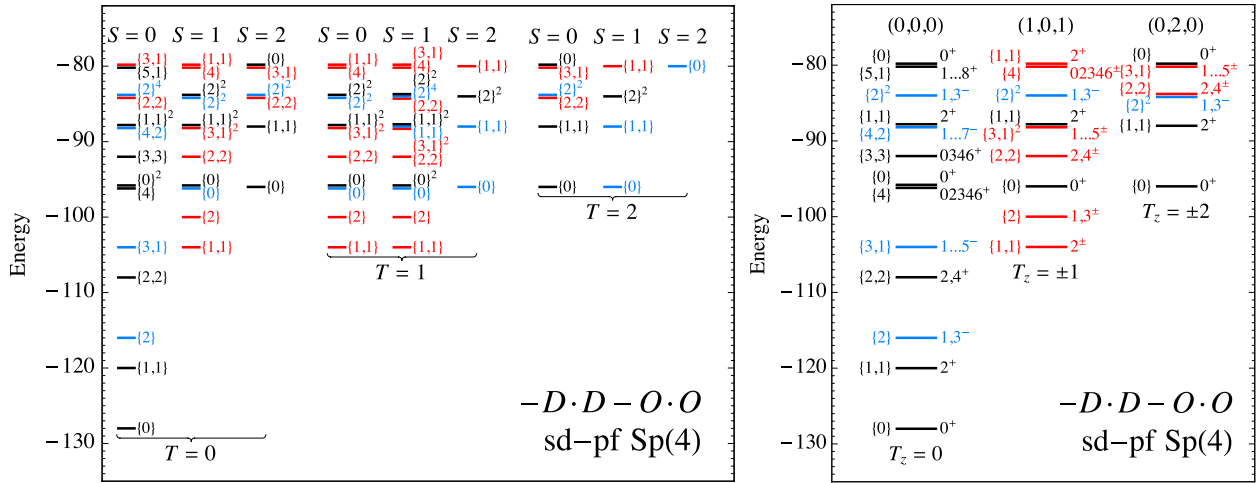


Fig. 7: Partial eigenspectrum of the hamiltonian  $-\hat{D} \cdot \hat{D} - \hat{O} \cdot \hat{O}$  for four nucleons in the  $sd$ - $pf$  shells. Levels of positive (negative) parity are black (blue) while parity doublets are red. Left panel: Levels are labelled by the  $Sp(4)$  quantum numbers  $\{\nu_1, \nu_2\}$  and values of the total spin  $S$  and the total isospin  $T$  are also indicated. If an irreducible representation  $\{\nu_1, \nu_2\}^k$  occurs  $k$  times for a given  $S$  and  $T$ , this is indicated by a superscript as  $\{\nu_1, \nu_2\}^k$ . Right panel: Only levels in the favoured supermultiplets are shown, which are  $(0, 0, 0)$  for  $T_z = 0$ ,  $(1, 0, 1)$  for  $T_z = \pm 1$  and  $(0, 2, 0)$  for  $T_z = \pm 2$ . Levels are labelled by the  $Sp(4)$  quantum numbers  $\{\nu_1, \nu_2\}$  on the left and by the orbital angular momenta  $L$  and parity  $\pi$  on the right. The supermultiplet labels  $(\lambda', \mu', \nu')$  and the isospin projection  $T_z$  are also indicated, and all levels have  $S = 0$ .

where  $\varphi$  is a phase. So one encounters the seemingly paradoxical situation that the hamiltonian (39), which does conserve parity, has eigenstates (40) that in general do not carry the parity quantum number. However, according to the energy formula (41) the states connected by the parity transformation, that is, states with the indices ‘ $a$ ’ and ‘ $b$ ’ interchanged, are degenerate in energy. Consequently, the states

$$\begin{aligned} & |[1^n]; ([\bar{h}]; [\bar{h}_a] \langle \bar{\omega}_a \rangle \times [\bar{h}_b] \langle \bar{\omega}_b \rangle; \langle \bar{\omega}_+ \rangle L) \times [\bar{h}'] ST]_{\pm} \\ & \equiv \frac{1}{\sqrt{2}} \left( 1 \pm \hat{P} \right) \\ & \times |[1^n]; ([\bar{h}]; [\bar{h}_a] \langle \bar{\omega}_a \rangle \times [\bar{h}_b] \langle \bar{\omega}_b \rangle; \langle \bar{\omega}_+ \rangle L) \times [\bar{h}'] ST]_{\pm}, \end{aligned}$$

are also eigenstates of the hamiltonian (39) and they do carry a definite parity quantum number,

$$\begin{aligned} & \hat{P} |[1^n]; ([\bar{h}]; [\bar{h}_a] \langle \bar{\omega}_a \rangle \times [\bar{h}_b] \langle \bar{\omega}_b \rangle; \langle \bar{\omega}_+ \rangle L) \times [\bar{h}'] ST]_{\pm} \\ & = \pm |[1^n]; ([\bar{h}]; [\bar{h}_a] \langle \bar{\omega}_a \rangle \times [\bar{h}_b] \langle \bar{\omega}_b \rangle; \langle \bar{\omega}_+ \rangle L) \times [\bar{h}'] ST]_{\pm}. \end{aligned}$$

These are the parity doublets that occur for a reflection-asymmetric quantum-mechanical system.

The eigenspectrum of the hamiltonian (39), including the parity of the levels, can now be determined. Figure 7 shows the example of four nucleons in the  $sd$ - $pf$  shells, in which case the hamiltonian reduces to  $-\hat{D} \cdot \hat{D} - \hat{O} \cdot \hat{O}$  and the symmetry is  $Sp(4)$ . The spectrum contains states (40) with identical indices ‘ $a$ ’ and ‘ $b$ ’,  $[\bar{h}_a] \{\bar{\nu}_a\} = [\bar{h}_b] \{\bar{\nu}_b\}$ , which have either positive or negative parity (black and blue, respectively, in figure 7). Furthermore, levels with  $[\bar{h}_a] \{\bar{\nu}_a\} \neq [\bar{h}_b] \{\bar{\nu}_b\}$  correspond to parity doublets (shown in red). The states shown in the left panel of figure 7 belong to all five possible supermultiplets and the spectrum is complete up to an energy  $E = -80$ . Retaining only the states contained in the favoured supermultiplets, we find the spectra shown in the right panel of figure 7 for  $T_z = 0, \pm 1$  and  $\pm 2$ . A striking feature of the eigenspectrum of the hamiltonian (39) is the presence of several parity dou-

plets at low excitation energy in the odd-odd system with  $T_z = \pm 1$ .

#### 4.3. The limit of large oscillator shells

While the components in equation (34) are associated with the orbital angular momentum, it is not *a priori* obvious that  $\hat{D}_\mu$  and  $\hat{O}_\mu$  have anything to do with the physical dipole and octupole operators. This relation is discussed in this subsection.

In second quantization any  $SU(4)$ -scalar operator of multipolarity  $\lambda$  can be written as

$$\hat{T}_\mu^{(\lambda)} = \sum_{\ell \ell'} t_{\ell \ell'}^{(\lambda)}(\mathcal{T}) \hat{G}_\mu^{(\lambda)}(\ell \ell'), \quad (42)$$

where the coefficients  $t_{\ell \ell'}^{(\lambda)}(\mathcal{T})$  can be considered as the definition of the operator  $\hat{T}_\mu^{(\lambda)}$ . In second quantization the multipole operator  $r^\lambda Y_{\lambda \mu}$  reads

$$\hat{T}_\mu(r^\lambda Y_\lambda) = -\sqrt{\frac{128\pi}{N+1}} \sum_{\ell \ell'} \frac{\langle \ell || r^\lambda Y_\lambda || \ell' \rangle}{\sqrt{2\lambda+1}} \hat{G}_\mu^{(\lambda)}(\ell \ell'), \quad (43)$$

where  $-\sqrt{128\pi/(N+1)}$  is a conventional factor introduced for reasons explained below. The choice of the potential well determines the coefficients  $t_{\ell \ell'}^{(\lambda)}(r^\lambda Y_\lambda)$  since the reduced matrix elements  $\langle \ell || r^\lambda Y_\lambda || \ell' \rangle$  depend on radial integrals. These are well known for the harmonic oscillator [28], leading to the coefficients

$$t_{\ell \ell'}^{(\lambda)}(r^\lambda Y_\lambda) = -(-)^\ell \sqrt{\frac{32}{N+1}} \hat{\ell} \hat{\ell}' \begin{pmatrix} \ell & \lambda & \ell' \\ 0 & 0 & 0 \end{pmatrix} I_{n \ell n' \ell'}^\lambda,$$

where  $I_{n \ell n' \ell'}^\lambda$  is the radial integral

$$I_{n \ell n' \ell'}^\lambda = \int_0^{+\infty} r^\lambda R_{n \ell}(r) R_{n' \ell'}(r) r^2 dr.$$

Alternatively, one may wish to consider the multipole operator  $Y_{\lambda\mu}$ , which in second quantization has the coefficients

$$t_{\ell\ell'}^{(\lambda)}(Y_\lambda) = -(-)^{\ell} \sqrt{\frac{32}{N+1}} \hat{\ell}\hat{\ell}' \begin{pmatrix} \ell & \lambda & \ell' \\ 0 & 0 & 0 \end{pmatrix},$$

which is identical to the expression for  $t_{\ell\ell'}^{(\lambda)}(r^\lambda Y_\lambda)$  but for the radial integral  $I_{n\ell n'\ell'}^\lambda$ .

As shown in section 3, Elliott's SU(3) model arises because the components of the orbital angular momentum operator (17), together with those of the quadrupole operator (18), close under commutation. No closure property is exactly valid for  $\hat{T}_\mu(r^\lambda Y_\lambda)$  if  $\lambda \neq 2$  but an approximate treatment is possible in the limit of large oscillator shells,  $N \rightarrow \infty$ . To see this point, we introduce into the generators (35) of  $\text{SOp}_-(\Omega)$  the explicit expressions (33) for the double tensors  $\hat{G}_{0\mu}^{(\lambda)}$  and  $\hat{G}_{\mu 0}^{(\lambda)}$ , leading to the coefficients

$$t_{\ell\ell'}^{(\lambda)}(T^{(\lambda)}) = -(-)^{\ell+N} \sqrt{32} \hat{\ell}\hat{\ell}' \left\{ \begin{matrix} \ell & \lambda & \ell' \\ \frac{1}{2}N & \frac{1}{2}N & \frac{1}{2}N \end{matrix} \right\}.$$

Since in the limit of large  $N$  one has the property [68]

$$\lim_{N \rightarrow \infty} (-)^N \sqrt{N+1} \left\{ \begin{matrix} \ell & \lambda & \ell' \\ \frac{1}{2}N & \frac{1}{2}N & \frac{1}{2}N \end{matrix} \right\} = \begin{pmatrix} \ell & \lambda & \ell' \\ 0 & 0 & 0 \end{pmatrix},$$

we conclude that the matrix elements of the generators  $\hat{T}_\mu^{(\lambda)}$  of  $\text{SOp}_-(\Omega)$  tend to those of the operators  $Y_{\lambda\mu}$ ,

$$\lim_{N \rightarrow \infty} t_{\ell\ell'}^{(\lambda)}(T^{(\lambda)}) = t_{\ell\ell'}^{(\lambda)}(Y_\lambda).$$

This approximation is reasonable even for relatively low values of  $N$  as long as  $\ell$  and  $\ell'$  are not too large. The factor  $r^\lambda$ , however, introduces a radial dependence (contained in the integrals  $I_{n\ell n'\ell'}^\lambda$ ) which is not included in the algebraic definition of the generators.

In table 2 are given the coefficients  $t_{\ell\ell'}^{(\lambda)}$  for three different kinds of dipole and octupole operators appropriate for the  $sd$ - $pf$  and  $pf$ - $sdg$  shells. The first line of each entry lists the coefficients for the generators  $\hat{T}_\mu^{(\lambda)}$  defined in equation (35). The second line gives the coefficients appropriate for the dipole or octupole operators  $Y_{\lambda\mu}$ . The normalization  $-\sqrt{32/(N+1)}$  is chosen such that the first coefficient,  $t_{sp}^{(1)}$  or  $t_{sf}^{(3)}$ , coincides with the algebraic definition in the first line. As shown above, the matrix elements of  $Y_{\lambda\mu}$  tend to those of the generators  $\hat{T}_\mu^{(\lambda)}$ , in the limit of large  $N \gg \ell, \ell'$ . It is indeed seen that deviations grow with increasing  $\ell$  and  $\ell'$ , especially for the octupole operator. The last line of each entry lists the coefficients appropriate for the dipole or octupole operators  $r^\lambda Y_{\lambda\mu}$  and calculated with harmonic-oscillator radial wave functions. Since the same normalization factor is adopted as for  $Y_{\lambda\mu}$ , the first coefficient,  $t_{sp}^{(1)}$  or  $t_{sf}^{(3)}$ , deviates from the algebraic definition and this deviation depends on the radial integrals  $I_{n\ell n'\ell'}^\lambda$ .

We conclude this subsection with two further remarks concerning the results in table 2. The first concerns the signs of the coefficients  $t_{\ell\ell'}^{(\lambda)}(Y_\lambda)$  and  $t_{\ell\ell'}^{(\lambda)}(r^\lambda Y_\lambda)$  which are seen in some cases to deviate from those of the coefficients of  $\hat{T}_\mu^{(\lambda)}$ . This can be remedied by a change of phase of the type

$$a_{\ell st}^\dagger \mapsto -a_{\ell st}^\dagger,$$

for certain single-particle orbital angular momenta  $\ell$ . Closure properties are not affected by such changes of phase and, consequently, the solvability property proven above remains valid. This property is well known for bosons [69, 70] and also applies to fermionic systems. For the  $sd$ - $pf$  and  $pf$ - $sdg$  shells enough freedom exists to accommodate any combination of signs of the coefficients  $t_{\ell\ell'}^{(\lambda)}$ . In other words, there always exists a change of phase of the  $a_{\ell st}^\dagger$  for certain single-particle orbital angular momenta  $\ell_i \in \{0, 1, \dots, N\}$  such that the signs of  $t_{\ell\ell'}^{(\lambda)}$  in the generators  $\hat{T}_\mu^{(\lambda)}$  are as required. This remains true for the dipole operator for any  $N$  but is no longer generally valid for the octupole operator if  $N \geq 5$ , that is, for the  $sdg$ - $pfh$  shells and beyond.

The second remark concerns the dependence of the coefficients  $t_{\ell\ell'}^{(\lambda)}(r^\lambda Y_\lambda)$  on the radial integrals  $I_{n\ell n'\ell'}^\lambda$ . The coefficients listed in table 2 are for a harmonic oscillator and other potentials lead to different results. Suppose we can find a potential with the property

$$I_{n\ell n'\ell'}^\lambda = c \frac{\left\{ \begin{matrix} \ell & \lambda & \ell' \\ \frac{1}{2}N & \frac{1}{2}N & \frac{1}{2}N \end{matrix} \right\}}{\begin{pmatrix} \ell & \lambda & \ell' \\ 0 & 0 & 0 \end{pmatrix}},$$

(where  $c$  is an arbitrary constant) for all  $(n, \ell, n', \ell')$  such that  $2n + \ell = 2n' + \ell' - 1 = N$ . For such a potential the following proportionality is valid without any approximation:

$$t_{\ell\ell'}^{(\lambda)}(T^{(\lambda)}) \propto t_{\ell\ell'}^{(\lambda)}(r^\lambda Y_\lambda),$$

implying an *exact* algebraic realization of the operator  $r^\lambda Y_\lambda$ .

#### 4.4. The octupole hamiltonian

The hamiltonian (39) contains the dipole interaction  $\hat{D} \cdot \hat{D}$ , which we may wish to eliminate. In particular, for  $N = 3$  and  $N = 4$  (the  $sd$ - $pf$  and  $pf$ - $sdg$  shells), two cases of main interest, elimination of  $\hat{D} \cdot \hat{D}$  leads to a pure octupole interaction  $\hat{O} \cdot \hat{O}$ . This elimination is achieved by noting that, since  $\hat{L}_\mu = \hat{L}_{a\mu} + \hat{L}_{b\mu}$  and  $\hat{D}_\mu = \hat{L}_{a\mu} - \hat{L}_{b\mu}$ , one has

$$\hat{D} \cdot \hat{D} = 2\hat{L}_a \cdot \hat{L}_a + 2\hat{L}_b \cdot \hat{L}_b - \hat{L} \cdot \hat{L}, \quad (44)$$

and therefore the dipole interaction is diagonal in the basis

$$\begin{array}{cccccc} \text{U}(\Omega^2) & \supset & \text{U}_a(\Omega) & \otimes & \text{U}_b(\Omega) & \supset & \text{SO}_{p_a}(\Omega) & \otimes & \text{SO}_{p_b}(\Omega) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ [\bar{h}] & & [\bar{h}_a] & & [\bar{h}_b] & & \langle \bar{\omega}_a \rangle & & \langle \bar{\omega}_b \rangle \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \supset & \text{SO}_a(3) & \otimes & \text{SO}_b(3) & \supset & \text{SO}_+(3) & & & \\ \downarrow & & \downarrow & & \downarrow & & & & \\ & L_a & & L_b & & L & & & \end{array}, \quad (45)$$

with eigenstates

$$|[1^n]; ([\bar{h}]; [\bar{h}_a] \langle \bar{\omega}_a \rangle L_a \times [\bar{h}_b] \langle \bar{\omega}_b \rangle L_b; L) \times [\bar{h}'] ST\rangle,$$

and with energy eigenvalues

$$2L_a(L_a + 1) + 2L_b(L_b + 1) - L(L + 1).$$

Table 2: Coefficients  $t_{\ell\ell'}^{(\lambda)}$  for dipole and octupole operators appropriate for the  $sd$ - $pf$  and  $pf$ - $sdg$  shells.

$N$	shells	operator	$t_{sp}^{(1)}$	$t_{sf}^{(3)}$	$t_{pd}^{(\lambda)}$	$t_{pg}^{(3)}$	$t_{df}^{(\lambda)}$	$t_{fg}^{(\lambda)}$
3	$sd$ - $pf$	$\hat{T}_\mu^{(1)}$	$2\sqrt{2}$	—	$8\sqrt{\frac{1}{5}}$	—	$2\sqrt{\frac{14}{5}}$	—
		$Y_{1\mu}$	$2\sqrt{2}$	—	4	—	$2\sqrt{6}$	—
		$rY_{1\mu}$	$2\sqrt{5}$	—	-4	—	$2\sqrt{21}$	—
		$\hat{T}_\mu^{(3)}$	—	$2\sqrt{2}$	$-2\sqrt{\frac{6}{5}}$	—	$4\sqrt{\frac{6}{5}}$	—
		$Y_{3\mu}$	—	$2\sqrt{2}$	$-6\sqrt{\frac{2}{7}}$	—	$-4\sqrt{\frac{2}{3}}$	—
		$r^3Y_{3\mu}$	—	$-3\sqrt{70}$	$6\sqrt{14}$	—	$-6\sqrt{21}$	—
4	$pf$ - $sdg$	$\hat{T}_\mu^{(1)}$	$4\sqrt{\frac{2}{5}}$	—	$2\sqrt{\frac{14}{5}}$	—	$8\sqrt{\frac{1}{5}}$	$4\sqrt{\frac{3}{5}}$
		$Y_{1\mu}$	$4\sqrt{\frac{2}{5}}$	—	$8\sqrt{\frac{1}{5}}$	—	$4\sqrt{\frac{6}{5}}$	$8\sqrt{\frac{2}{5}}$
		$rY_{1\mu}$	$-8\sqrt{\frac{1}{5}}$	—	$4\sqrt{\frac{14}{5}}$	—	$-4\sqrt{\frac{6}{5}}$	$24\sqrt{\frac{1}{5}}$
		$\hat{T}_\mu^{(3)}$	—	$4\sqrt{\frac{2}{5}}$	$-8\sqrt{\frac{3}{35}}$	$12\sqrt{\frac{1}{35}}$	$4\sqrt{\frac{3}{35}}$	$6\sqrt{\frac{22}{35}}$
		$Y_{3\mu}$	—	$4\sqrt{\frac{2}{5}}$	$-12\sqrt{\frac{2}{35}}$	$8\sqrt{\frac{6}{35}}$	$-8\sqrt{\frac{2}{15}}$	$-24\sqrt{\frac{1}{55}}$
		$r^3Y_{3\mu}$	—	$12\sqrt{\frac{14}{5}}$	$-78\sqrt{\frac{1}{5}}$	$-36\sqrt{\frac{6}{5}}$	$24\sqrt{\frac{6}{5}}$	$-18\sqrt{\frac{22}{5}}$

The bases (29) and (45) are connected by a unitary transformation,

$$\begin{aligned}
 & |\langle \bar{\omega}_a \rangle \times \langle \bar{\omega}_b \rangle; \langle \bar{\omega}_+ \rangle L \rangle \\
 &= \sum_{L_a L_b} \left\langle \begin{array}{cc} \langle \bar{\omega}_a \rangle & \langle \bar{\omega}_b \rangle \\ L_a & L_b \end{array} \middle| \begin{array}{c} \langle \bar{\omega}_+ \rangle \\ L \end{array} \right\rangle |\langle \bar{\omega}_a \rangle L_a \times \langle \bar{\omega}_b \rangle L_b; L \rangle,
 \end{aligned}$$

where the symbol in angle brackets is an isoscalar factor [71], associated with either  $\text{Sp}(\Omega) \supset \text{SO}(3)$  if  $\Omega$  is even or  $\text{SO}(\Omega) \supset \text{SO}(3)$  if  $\Omega$  is odd. Note that this transformation does not depend on other labels of the states (29) and (45), which are therefore suppressed. The combination of the previous results leads to the following expression for the matrix elements of  $\hat{D} \cdot \hat{D}$ :

$$\begin{aligned}
 & \langle \langle \bar{\omega}_a \rangle \times \langle \bar{\omega}_b \rangle; \langle \bar{\omega}_+ \rangle L | \hat{D} \cdot \hat{D} | \langle \bar{\omega}_a \rangle \times \langle \bar{\omega}_b \rangle; \langle \bar{\omega}_+ \rangle L \rangle \\
 &= \sum_{L_a L_b} [2L_a(L_a + 1) + 2L_b(L_b + 1) - L(L + 1)] \\
 & \times \left\langle \begin{array}{cc} \langle \bar{\omega}_a \rangle & \langle \bar{\omega}_b \rangle \\ L_a & L_b \end{array} \middle| \begin{array}{c} \langle \bar{\omega}_+ \rangle \\ L \end{array} \right\rangle \left\langle \begin{array}{cc} \langle \bar{\omega}_a \rangle & \langle \bar{\omega}_b \rangle \\ L_a & L_b \end{array} \middle| \begin{array}{c} \langle \bar{\omega}_+ \rangle \\ L \end{array} \right\rangle.
 \end{aligned}$$

The octupole interaction  $\hat{O} \cdot \hat{O}$  is not diagonal in the basis (40) but rather block diagonal. The blocks consist of basis states that have different labels  $\langle \bar{\omega}_+ \rangle$  but otherwise identical quantum numbers. Provided the necessary isoscalar factors are known, the determination of the eigenspectrum of  $\hat{O} \cdot \hat{O}$  only requires the diagonalization of matrices of modest size (dimension  $\sim 10$ ).

## 5. Concluding remarks

We have discussed in this paper two analytic solutions of the spherical shell model that make contact with the geometric collective model. The first is based on Elliott's well-known  $\text{SU}(3)$  symmetry, which is generated by a spin-isospin-scalar quadrupole interaction  $r^2Y_2 \cdot r^2Y_2$  and provides a natural explanation of the phenomenon of nuclear rotation. A prerequisite for its existence is the spin-isospin  $\text{SU}(4)$  symmetry of Wigner's supermultiplet model. Only

when the condition of exact solvability is relaxed can one propose extensions such as pseudo- $\text{SU}(3)$  or quasi- $\text{SU}(3)$  that apply to more realistic  $jj$ -coupled situations.

We have shown in this paper that another analytic solution of the spherical shell model exists, generated by a spin-isospin-scalar octupole interaction, which in the limit of large oscillator shells tends to its geometric equivalent  $Y_3 \cdot Y_3$ . A natural outcome of the ensuing symmetry, which can be either orthogonal or symplectic, is the presence in the excitation spectrum of parity doublets, as is required of a reflection-asymmetric quantum-mechanical object.

The octupole solution of the spherical shell model presented in this paper is developed to a schematic level only and it is too early to tell whether it can be extended to more realistic situations with applications to actual nuclei. Among the open problems that need to be explored are the non-degeneracy of the lower and upper oscillator shells, and the departure from spin-isospin symmetry (*e.g.*, the spin-orbit interaction). Furthermore, in the heavy regions of the periodic table, where octupole deformation is relevant, the neutron and proton orbitals are different, though for both it is possible to choose neighboring shells with different parities. Also, the questions of possible effects resulting from the differences between the algebraic and geometric octupole operators as well as the elimination of states due to spurious centre-of-mass motion need to be addressed.

The existence of an octupole solution of the spherical shell model, as an alternative to Elliott's quadrupole solution, also raises the question whether the two deformations can be combined into one model. It may not be possible to elaborate a fully solvable quadrupole-octupole shell-model hamiltonian but recent studies indicate that the application range of dynamical symmetries is considerably extended through partial solvability [72]. It will therefore be of interest to apply this notion to the two deformed symmetries of the spherical shell model discussed in this paper.

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## Appendix: Branching rules

The applications presented in this paper require the knowledge of a variety of branching rules, the problem of which can be formulated as follows. Given two algebras  $G_1$  and  $G_2$  with  $G_1 \supset G_2$ , what irreducible representations of  $G_2$  are contained in a given irreducible representation of  $G_1$ ? Note that, if  $G_2 = \text{SO}(3)$ , this amounts to finding the (orbital) angular momentum content of a given irreducible representation of  $G_1$ .

The branching rules associated with unitary algebras concern the following cases:

- $U(n_1) \supset U(n_2)$ ,  $n_1 > n_2$ ,
- $U(n_1 n_2) \supset U(n_1) \otimes U(n_2)$ ,
- $U(n_1 + n_2) \supset U(n_1) \otimes U(n_2)$ .

These, of course, are known since long from the classical theory of group characters and representations, as described, for example, in the monographs by Murnaghan [32] and Littlewood [33]. The purpose of this appendix is to show that all such branching rules can be derived from a few known results concerning Young diagrams that can be programmed in a symbolic language like *Mathematica* [73].

The first result concerns the (outer) multiplication of two Young diagrams  $[\bar{h}_1]$  and  $[\bar{h}_2]$ , which can be written as

$$[\bar{h}_1] \times [\bar{h}_2] = \sum_{\bar{h}} \Gamma_{\bar{h}_1 \bar{h}_2 \bar{h}} [\bar{h}],$$

where  $\Gamma_{\bar{h}_1 \bar{h}_2 \bar{h}}$  denotes the number of times  $(0, 1, 2, \dots)$   $[\bar{h}]$  occurs in the outer product  $[\bar{h}_1] \times [\bar{h}_2]$ . The rules for calculating such outer products of Young diagrams can be found in many textbooks (see, for example, section 7-12 of Ref. [34]) and will not be repeated here.

The second result states that any Young diagram can be written as a linear combination of products of symmetric Young diagrams. The explicit decomposition for a Young diagram of length  $s$ ,  $[\bar{h}] \equiv [h_1, h_2, \dots, h_s]$ , reads

$$[\bar{h}] = \begin{vmatrix} [h_1] & [h_1 + 1] & \cdots & [h_1 + s - 1] \\ [h_2 - 1] & [h_2] & \cdots & [h_2 + s - 2] \\ \vdots & \vdots & \ddots & \vdots \\ [h_s - s + 1] & [h_s - s + 2] & \cdots & [h_s] \end{vmatrix},$$

with the convention that  $[h] = 1$  for  $h = 0$  and  $[h] = 0$  for  $h < 0$ . The most elementary example is the decomposition of  $[1, 1]$ , for which  $h_1 = h_2 = 1$  and  $s = 2$ , and therefore

$$[1, 1] = \begin{vmatrix} [1] & [2] \\ 1 & [1] \end{vmatrix} = [1] \times [1] - [2].$$

In general, we write the decomposition of  $[\bar{h}]$  as

$$[\bar{h}] = \sum_{\{r_1, r_2, \dots, r_k\}} a_{r_1 r_2 \dots r_k}^{[\bar{h}]} \prod_{i=1}^k [r_i],$$

with products  $[r_1] \times [r_2] \times \cdots \times [r_k]$  of one-rowed Young diagrams of lengths  $r_1, r_2, \dots, r_k$  and coefficients  $a_{r_1 r_2 \dots r_k}^{[\bar{h}]}$  that are readily obtained from the determinantal expression.

To obtain branching rules associated with the reduction  $G_1 = U(n) \supset G_2$ , one needs to specify what is the branching rule for the fundamental representation  $[1]$  of  $U(n)$ . This rule can be written generically as  $[1] \mapsto [\bar{f}]$ , where  $[\bar{f}]$  is an irreducible representation of  $G_2$ . For example, in the reduction  $U(\Gamma) \supset U(3)$  of Elliott's model for the oscillator shell with major quantum number  $N$  [and  $\Gamma \equiv (N+1)(N+2)/2$ ], the fundamental representation  $[1]$  of  $U(\Gamma)$  reduces to  $[N, 0, 0]$  of  $U(3)$ . In this case therefore  $[\bar{f}] = [N, 0, 0]$ , which expresses the fact that one nucleon in the  $N$ -shell corresponds to  $N$  oscillator quanta. Given the property  $[1] \mapsto [\bar{f}]$  in the reduction  $U(n) \supset G_2$ , the branching rule for a general irreducible representation  $[\bar{h}]$  of  $U(n)$  can be written as

$$[\bar{h}] \mapsto [\bar{f}] \odot [\bar{h}],$$

where  $\odot$  is the operation of 'plethysm', whose properties are discussed below. On the left-hand side of  $\mapsto$  stands the irreducible representation  $[\bar{h}]$  of  $U(n)$  while on the right-hand side stands a sum of irreducible representations of  $G_2$ . The latter is found by carrying out the plethysm, which can be done by applying the following rules:

$$\begin{aligned} [\bar{f}] \odot [1] &= [\bar{f}], \\ [\bar{f}] \odot ([\bar{h}_1] \pm [\bar{h}_2]) &= [\bar{f}] \odot [\bar{h}_1] \pm [\bar{f}] \odot [\bar{h}_2], \\ [\bar{f}] \odot ([\bar{h}_1] \times [\bar{h}_2]) &= ([\bar{f}] \odot [\bar{h}_1]) \times ([\bar{f}] \odot [\bar{h}_2]), \end{aligned}$$

where  $\times$  refers to an outer multiplication in  $G_2$ . The first rule is nothing but the definition of the branching rule for the fundamental representation while the successive application of the second and third rules lead to the result

$$[\bar{h}] \mapsto \sum_{\{r_1, r_2, \dots, r_k\}} a_{r_1 r_2 \dots r_k}^{[\bar{h}]} \prod_{i=1}^k ([\bar{f}] \odot [r_i]),$$

where the product refers to outer multiplication in  $G_2$ . This shows that the problem of working out the general plethysm  $[\bar{f}] \odot [\bar{h}]$  is reduced to that of finding  $[\bar{f}] \odot [h]$ , where the irreducible representation  $[h]$  of  $U(n)$  is symmetric and characterized by a one-rowed Young diagram.

For the branching rule associated with the reduction  $U(n_1 n_2) \supset U(n_1) \otimes U(n_2)$  the following result is valid for the symmetric representation  $[h]$  of  $U(n_1 n_2)$ :

$$[h] \mapsto \sum_{\bar{s}} [\bar{s}] \otimes [\bar{s}],$$

where the sum is over all Young diagrams with  $h$  boxes and at most  $\min(n_1, n_2)$  rows,  $s_1 + s_2 + \cdots + s_k = h$  and  $k \leq \min(n_1, n_2)$ . We do not give a formal proof of this result but it is clear on intuitive grounds since, for a symmetric wave function, the symmetry character under the partial exchange of coordinates must be compensated by an equivalent symmetry character under the exchange of the remainder of the coordinates.

For the branching rule associated with the reduction  $U(n_1 + n_2) \supset U(n_1) \otimes U(n_2)$  the following result is valid

for the symmetric representation  $[h]$  of  $U(n_1 + n_2)$ :

$$[h] \mapsto \sum_{r=0}^h [h-r] \otimes [r].$$

Again we do not give a formal proof of the result, which in this case follows from the conservation of particle number. Note that representations of  $U(n_1) \otimes U(n_2)$  are written as  $[\bar{s}_1] \otimes [\bar{s}_2]$  with the understanding that  $[\bar{s}_i]$  is associated with  $U(n_i)$ .

With these results the second and third of our list of unitary branching rules are determined in complete generality and the only remaining task is to find the branching rule associated with  $U(n_1) \supset U(n_2)$  for a symmetric representation  $[h]$  of  $U(n_1)$ . Let us introduce the notation

$$[\bar{f}] \odot [h] = \sum_{\bar{s}} x_{\bar{s}} [\bar{s}],$$

with coefficients  $x_{\bar{s}} = 0, 1, 2, \dots$  that are to be determined. As can be understood from the general rules governing plethysms, the sum is over all Young diagrams  $[\bar{s}]$  with  $f$  boxes, where  $f$  is the number of boxes of the fundamental representation  $[\bar{f}]$ . Furthermore, we denote by  $[\bar{s}]/[1]$  all permissible Young diagrams that can be obtained by deleting one box from  $[\bar{s}]$  or, explicitly,

$$[\bar{s}]/[1] \equiv \sum_{\bar{u}} \Gamma_{[1]\bar{u}\bar{s}} [\bar{u}].$$

Littlewood's third method [33, 47] states that the coefficients  $x_{\bar{s}}$  are solutions of the equation

$$\sum_{\bar{s}} x_{\bar{s}} ([\bar{s}]/[1]) = ([\bar{f}] \odot [h-1]) \times ([\bar{f}]/[1]),$$

which, if the fundamental representation has only one row,  $[\bar{f}] = [f]$ , reduces to

$$\sum_{\bar{s}} x_{\bar{s}} ([\bar{s}]/[1]) = ([f] \odot [h-1]) \times [f-1].$$

Since  $[\bar{f}] \odot [1] = [\bar{f}]$ , the series on the right-hand side can be generated by an induction hypothesis and subsequent outer multiplications in  $U(n_2)$ . The coefficients  $x_{\bar{s}}$  must be chosen such that all terms on left- and right-hand sides are identical. This condition is not always sufficient to determine all coefficients  $x_{\bar{s}}$  uniquely and must be supplemented by the equation

$$\sum_{\bar{s}} d_{[\bar{s}]}(n_2) x_{[\bar{s}]} = d_{[h]}(n_1),$$

where  $d_{[\bar{s}]}(n)$  is the dimension of the irreducible representation  $[\bar{s}]$  in  $U(n)$ . In all applications presented in this paper we have found that the combination of the two conditions suffices to determine the plethysm  $[\bar{f}] \odot [h]$ , and therefore the general plethysm  $[\bar{f}] \odot [\bar{h}]$  for the reduction  $U(n_1) \supset U(n_2)$ .

The branching rules associated with the restricted algebras concern the following cases:

- $U(n) \supset SO(n)$ ,
- $U(n) \supset Sp(n)$ .

These rules can be taken from Littlewood [33] and read in our notation

$$\begin{aligned} [\bar{h}] &\mapsto \langle \bar{h} \rangle + \sum_{\bar{\delta}} \Gamma_{\bar{\delta}\bar{v}\bar{h}} \langle \bar{v} \rangle, \\ [\bar{h}] &\mapsto \{ \bar{h} \} + \sum_{\bar{\beta}} \Gamma_{\bar{\beta}\bar{v}\bar{h}} \{ \bar{v} \}, \end{aligned}$$

for  $U(n) \supset SO(n)$  and  $U(n) \supset Sp(n)$ , respectively. The  $\Gamma$ s refer to outer multiplication in  $U(n)$  and the summations are over the partitions

$$\begin{aligned} [\bar{\delta}] &= [2], [4], [2^2], [6], [4, 2], [2^3], \dots, \\ [\bar{\beta}] &= [1^2], [2^2], [1^4], [3^2], [2^2, 1^2], [1^6], \dots, \end{aligned}$$

that is, all partitions constructed out of  $[2]$  and  $[1^2]$ , respectively. Some of the irreducible representations of  $SO(n)$  and  $Sp(n)$  that appear in these branching rules have more labels than allowed and must be eliminated or modified. The appropriate modification rules are given by Wybourne [47].

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