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NEUTRON–PROTON PAIRS IN NUCLEI

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A review is given of attempts to describe nuclear properties in terms of neutron–proton pairs that are subsequently replaced by bosons. Some of the standard approaches with low-spin pairs are recalled but the emphasis is on a recently proposed framework with pairs of neutrons and protons with aligned angular momentum. The analysis is carried out for general j and applied to $N = Z$ nuclei in the $1f_{7/2}$ and $1g_{9/2}$ shells.

Keywords: shell model; interacting boson model; nucleon pairs.

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1. Introduction

In dealing with complex systems with many elementary components, one of the major goals of physics is to seek simplifications by adopting a description in terms of composite structures. An obvious example of this approach is found in nuclear physics, when the elementary constituents of the nucleus, quarks, are lumped into nucleons—an approximation adequate for the description of most nuclear phenomena at low energy. Still, the nuclear many-body problem in terms of nucleons instead of quarks is fiendishly difficult to solve for all but the lightest of nuclei, and further simplifying assumptions are required for the majority of them. One possibility is to lump the nucleons into pairs and attempt a description of nuclear phenomena in terms of those.

While such nucleon-pair models can be simple and attractive in principle, their success obviously depends on the type of pairs considered. This choice should be guided by nature of the interaction between the nucleons. One of the defining features of the nuclear force is that it is strongly attractive between nucleons that are paired to angular momentum $J = 0$. Models where the pairing component of the interaction is prominent therefore have played an important part in the development of our understanding of nuclear structure.¹ While pairs consisting of *identical* nucleons (*i.e.*, neutron–neutron or proton–proton) are by now an accepted feature of nuclei, much debate still exists concerning the role of neutron–proton pairs. One

component of neutron–proton pairing is of isovector character, and arguments of isospin symmetry require that it should be considered on the same footing as its neutron–neutron or proton–proton equivalent. A neutron and a proton can also interact via an isoscalar component of the nuclear force and the debate is whether pairing of this type leads to enhanced collectivity and correlated states. This question is still unanswered after several decades of research.^{2–8} A recent review of possible signatures of isoscalar neutron–proton pairing is given by Macchiavelli.⁹

This paper certainly does *not* give a comprehensive and exhaustive review of the role of neutron–proton pairs in nuclei. Rather, it zooms in on a particular approach which replaces pairs of nucleons by bosons—approximation known under the name of ‘interacting boson model’—and within this class of models attention is paid to those that adopt bosons that stem from neutron–proton pairs. The standard boson models of this kind are briefly described in Sect. 2 but the emphasis is on a recently proposed framework with bosons that correspond to neutron–proton pairs with aligned, high angular momentum. The motivation for and the historical context of this new approach are outlined in Sect. 3. The main purpose of the present review is to argue that the proper framework to develop this approach is by applying boson mapping techniques to the nucleon-pair shell model. Technical aspects are reviewed in Sects. 4 and 5 while Sect. 6 gives a non-technical summary of the various approximations that enter a description in terms of aligned neutron–proton pairs or bosons. Applications to the $1f_{7/2}$ and $1g_{9/2}$ shells are discussed in Sect. 7. In fact, most of the results shown in that section are new and in this sense the present paper is not a review of published research. Nevertheless, it is the opinion of the author that they clarify the issue of the role in nuclei of neutron–proton pairs, aligned or otherwise. Finally, conclusions are drawn in Sect. 8.

2. Standard boson models with neutron–proton pairs: IBM-3 and IBM-4

The interacting boson model (IBM) of Arima and Iachello¹⁰ starts from the premise that low-lying collective excitations can be described in terms of nucleon pairs (with angular momentum 0 and 2, in the most elementary version of the model) and that these pairs can be approximated as (s and d) bosons. If neutrons and protons occupy different valence shells, it is natural to consider neutron–neutron and proton–proton pairs only, and to include the neutron–proton interaction as a force between the two types of pairs. This then leads to a version of the IBM with two kinds of bosons,¹¹ of neutron and of proton type, the so-called IBM-2. If neutrons and protons occupy the same valence shell, this approach is no longer valid since there is no reason not to include a neutron–proton pair with isospin $T = 1$. The version of the IBM that also contains the $T = 1$ neutron–proton boson, proposed by Elliott and White,¹² is called IBM-3. Because the IBM-3 includes the complete $T = 1$ triplet, it can be made isospin invariant, enabling the construction of states with good total angular momentum J and good total isospin T and leading therefore to a more direct

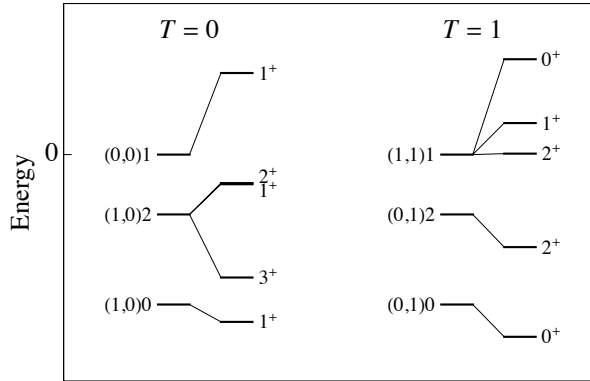


Fig. 1. The energy spectrum of a neutron and a proton in a p shell, interacting through an attractive delta force. States with isospin $T = 0$ ($T = 1$) are shown on the left (right). Levels are labelled on the left by $(S, T)L$, the spin S , the isospin T and the orbital angular momentum L , and on the right by the angular momentum and parity J^π . All these quantum numbers are conserved in the absence of a spin-orbit splitting (left) while only the J^π symmetry remains for a non-zero spin-orbit splitting (right).

comparison with the shell model (see, *e.g.*, Ref. 13).

The bosons of the IBM-3 all have isospin $T = 1$ and, in principle, other bosons can be introduced, in particular those that correspond to $T = 0$ neutron-proton pairs. This further extension (proposed by Elliott and Evans¹⁴ and referred to as IBM-4) is the most elaborate version of the standard IBM. The bosons are assigned an orbital angular momentum L , a spin S and an isospin T , and in IBM-4 the choice $L = 0$ and $L = 2$ is retained with either $(S, T) = (0, 1)$ or $(S, T) = (1, 0)$. The total angular momentum J of the bosons is obtained by coupling L and S , leading to an ensemble of bosons with $(J, T) = (0, 1)$, $(2, 1)$, $(1, 0)^2$, $(2, 0)$ and $(3, 0)$.

The justification of this particular choice of bosons is based on the shell model. Consider as an example a neutron and a proton in a p shell. The effective force between the two nucleons is of a short-range nature and can, within a reasonable approximation, be represented as an attractive delta interaction, $\hat{V}(\bar{r}_1 - \bar{r}_2) = -g\delta(\bar{r}_1 - \bar{r}_2)$ with $g > 0$. Under the assumption of zero spin-orbit splitting (*i.e.*, degenerate $p_{1/2}$ and $p_{3/2}$ shells), the energy spectrum can be worked out on the basis of simple symmetry arguments (see Fig. 1). Since the interaction is spin and isospin independent, the LS coupling scheme applies and all states can be assigned an orbital angular momentum L , a spin S and an isospin T . Furthermore, because of overall anti-symmetry, all states are characterized by either spatial symmetry ($L = 0$ or 2) and spin-isospin anti-symmetry [$(S, T) = (0, 1)$ or $(1, 0)$], or spatial anti-symmetry ($L = 1$) and spin-isospin symmetry [$(S, T) = (0, 0)$ or $(1, 1)$]. The former states are lowered in energy by the attractive delta force ($L = 0$ more so than $L = 2$) while the interaction energy in the latter states is exactly zero because of their spatial anti-symmetry. The states lowered in energy are precisely

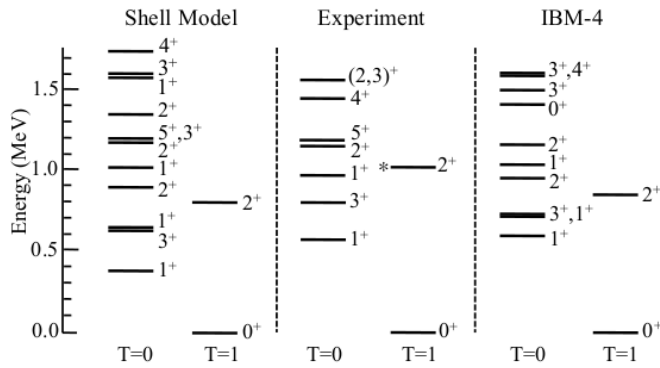


Fig. 2. The spectrum of states in ^{62}Ga with isospin $T = 0$ and $T = 1$. Levels are labelled by their angular momentum and parity J^π . The different columns contain the results of a large-scale shell-model calculation,²² of a mapped IBM-4 calculation²³ and the experimental levels.²⁴ Figure taken from David *et al.*²⁴

those that correspond to the bosons in IBM-4. For a realistic choice of spin-orbit splitting, the many degeneracies are lifted, lowering the higher- J levels in energy (see Fig. 1). The choice of bosons in IBM-4 allows a classification where states carry the quantum numbers of total orbital angular momentum L , total spin S , total angular momentum J and total isospin T , in addition to the $SU(4)$ labels (λ, μ, ν) of Wigner's supermultiplet scheme,¹⁵ in close analogy with the corresponding shell-model labels.

These qualitative arguments in favour of the IBM-4 have been corroborated by quantitative, microscopic studies in even-even¹⁶ and odd-odd¹⁷ sd -shell nuclei. In heavier nuclei the situation is more complex. The effect of the spin-orbit force is such that the LS -coupling scheme no longer applies, resulting in the breaking of the L and S quantum numbers, in contrast to the total angular momentum J which is of course exactly conserved because of rotational invariance and the total isospin T which conserved to a good approximation. Nevertheless, the L and S quantum numbers of the shell model can be replaced by their 'pseudo' equivalents, along the original ideas of Hecht and Adler,¹⁸ and Arima *et al.*¹⁹ This might be possible in specific regions of the nuclear chart²⁰ and is borne out by shell-model calculations with realistic interactions in nuclei beyond ^{56}Ni .²¹ The existence of these approximate symmetries in the shell model allows a mapping onto IBM-4. A typical example is provided by the $N = Z$ nucleus ^{62}Ga . The spectroscopy predicted in the shell model, with a space consisting of the orbits $2p_{3/2}$, $1f_{5/2}$, $2p_{1/2}$ and $1g_{9/2}$, is very complex with intertwined states of isospin $T = 0$ and $T = 1$.²² Most levels are of low spin and those are well reproduced in the mapped IBM-4 calculation.²³ It is also found, however, that $T = 0$ levels with higher spin (5^+ , 6^+ and 7^+) are at significantly higher energies in the IBM-4 or even absent from it. This result is not surprising since the standard IBM-4 choice consists of bosons with rather low spin

(up to $J^\pi = 3^+$). In a recent experiment, David *et al.*²⁴ have observed a number of additional levels of low spin, presumably with isospin $T = 0$, as predicted by the shell model and the IBM-4 (see Fig. 2). It remains nevertheless true that states with higher spin require an approach which is different from the standard IBM-4.

3. Aligned neutron–proton pairs

In a recent paper, Cederwall *et al.*²⁵ propose an alternative description of $N \sim Z$ nuclei in terms of neutron–proton pairs with *aligned* spin, henceforth referred to as B pairs. The proposal concerns massive $N \sim Z$ nuclei, such as ^{92}Pd , approaching ^{100}Sn , with valence nucleons dominantly in the $1g_{9/2}$ shell. The claim is made (see also Refs. 26 and 27) that low-lying yrast states in ^{92}Pd and neighbouring nuclei are mainly built out of aligned neutron–proton isoscalar (with isospin $T = 0$) pairs with angular momentum $J = 9$. For the purpose of constructing a boson model, the aligned-pair scheme is particularly attractive since it involves a single neutron–proton pair; if valid, the many bosons of IBM-4 can be replaced by a single one.

Related ideas have been explored in the past. One is the stretch scheme of Danos and Gillet^{28,29} which applies to even–even $N = Z$ nuclei. It assumes that half of the neutrons align with half of the protons to form a state of maximum angular momentum. Similarly, the other half of the nucleons aligns to a state with the same angular momentum. The total angular momentum of the system is generated by the coupling of these two fixed stretched configurations. For four nucleons the stretch scheme is exactly equivalent to the description in terms of aligned pairs as proposed by Blomqvist and co-workers.²⁵ For eight, twelve, . . . nucleons, however, the stretch scheme is different since any angular momentum is uniquely defined in terms of the two stretch configurations while it generally can be written in several ways in terms of B pairs. As a result, the stretch scheme has less flexibility to provide an adequate approximation of a realistic shell-model wave function. An explicit relation between both approximations is established in Sect. 7.

Blomqvist’s aligned B pairs are in fact identical to the ‘ q pairs’ introduced in the 1980s by Daley. A q -pair analysis of the $1f_{7/2}$ shell with a schematic delta interaction exists as a Daresbury preprint³⁰ but, unfortunately, not as a published paper. The study of Daley concentrates on the even–even nuclei ^{44}Ti and ^{48}Cr , and only in the former nucleus does he find results similar to the ones shown below. No analysis of odd–odd nuclei is presented.

A crucial issue in any model that represents a fermionic system in terms of pairs (or, more generally, clusters) of fermions is the representation of exchange effects resulting from the Pauli principle as interactions between these clusters. In the stretch scheme of Danos and Gillet^{28,29} anti-symmetry between the two stretched configurations is neglected while it is not clear from Daley’s paper³⁰ to what extent the interactions between his q bosons include Pauli effects. On the other hand, anti-symmetry is fully taken into account in the multi-step shell-model approach of Qi *et al.*,²⁶ at the expense of major computational complexities which hinder

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an easy, intuitive interpretation of the results. One of the aims of this review is to analyze results of shell-model calculations in terms of B pairs with the nucleon-pair shell model. Although numerically challenging, in particular for B pairs in view of their high angular momentum, this approach provides a conceptually simple way to treat Pauli exchange effects between the pairs and subsequently represent those as interactions between bosons. The technical aspects of this approach are reviewed in the next two sections.

4. Nucleon-pair shell model

The natural framework to test Blomqvist's truncation idea is provided by the nucleon-pair shell model (NPSM).^{31,32} In the NPSM a basis is constructed from nucleon pairs. These can be collective superpositions of two-particle states or they may be identified with pure two-particle states themselves. The many applications of this formalism are reviewed in Ref. 33. The extension of the NPSM that includes isospin³⁴ is of particular relevance here.

In the language and notation of the NPSM, Blomqvist's idea can be summarized with the statement that the full $T = 0$ shell-model space can be reduced to one constructed out of aligned neutron-proton pairs of which the basis states are written as

$$|B^n L_2 \dots L_n\rangle \equiv \left(\dots \left((B^\dagger \times B^\dagger)^{(L_2)} \times B^\dagger \right)^{(L_3)} \times \dots \times B^\dagger \right)^{(L_n)} |o\rangle, \quad (1)$$

with $|o\rangle$ the vacuum state. Pairs with angular momentum J and projection M_J , and with isospin T and projection M_T are denoted by

$$P_{JM_T M_T}^\dagger \equiv (a_{jt}^\dagger \times a_{jt}^\dagger)_{M_J M_T}^{(JT)}, \quad (2)$$

where $a_{jm_j t m_t}^\dagger$ creates a nucleon with angular momentum j and projection m_j , and isospin $t = \frac{1}{2}$ and projection m_t . The short-hand notation $B_{M_J}^\dagger$ (B for Blomqvist) is used in Eq. (1) for a creation operator of a neutron-proton pair with angular momentum $J = 2j$ and isospin $T = 0$. The $2n$ -particle state (1) is characterized by the intermediate angular momenta L_2, \dots, L_n , where L_n is the final and total angular momentum of the state. In the basis (1) all pairs have $T = 0$ and the coupling in isospin need not be considered.

The basis (1) is non-orthogonal and possibly overcomplete. Any calculation in this basis must therefore start from the diagonalization of the overlap matrix

$$\langle B_i^n | B_{i'}^n \rangle \equiv \langle B^n L_2 \dots L_{n-1} L_n | B^n L'_2 \dots L'_{n-1} L_n \rangle, \quad (3)$$

where in bra and ket of the matrix element all possible intermediate angular momenta L_2, \dots, L_{n-1} must be considered, leading to a series of basis states denoted as $|B_i^n\rangle$ where i is a short-hand notation for the set $\{L_2, \dots, L_{n-1}\}$. The computation of the matrix elements (3) is complicated but possible with the recurrence relation devised by Chen.³² Vanishing eigenvalues of the overlap matrix $\langle B_i^n | B_{i'}^n \rangle$

indicate the overcompleteness of the pair basis. If a selection of ω pair-basis states is made for which all eigenvalues of the overlap matrix are non-zero, the following linear combinations can be constructed:

$$|\bar{B}_k^n\rangle = \sqrt{\frac{1}{o_k}} \sum_{i=1}^{\omega} c_{ki} |B_i^n\rangle, \quad k = 1, \dots, \omega, \quad (4)$$

where o_k is the k^{th} eigenvalue of the overlap matrix and c_{ki} ($i = 1, \dots, \omega$) the associated eigenvector. The vectors $|\bar{B}_k^n\rangle$ ($k = 1, \dots, \omega$) are normalized, orthogonal and linearly independent, and therefore provide a proper basis for a shell-model calculation, albeit a truncated one. For a given shell-model hamiltonian \hat{H}^f , the energy spectrum and eigenvectors can be obtained from the diagonalization of the matrix

$$\langle \bar{B}_k^n | \hat{H}^f | \bar{B}_{k'}^n \rangle = \sqrt{\frac{1}{o_k o_{k'}}} \sum_{i, i'=1}^{\omega} c_{ki} c_{k'i'} \langle B_i^n | \hat{H}^f | B_{i'}^n \rangle, \quad k, k' = 1, \dots, \omega. \quad (5)$$

The formalism as explained so far allows one to perform a shell-model calculation in a truncated basis constructed from aligned $T = 0$ neutron-proton pairs. In subsequent applications we will also want to analyze arbitrary shell-model wave functions in terms of B pairs. An analysis of this type clearly cannot be carried out in the basis (1)—since the latter spans only part of the shell-model space—and it requires a generalization to a basis in terms of arbitrary pairs. The formalism of the NPSM with isospin, needed to this end, is detailed in Ref. 34 and only a few basic formulas are given here.

It is convenient to introduce the following short-hand notation for the pairs:

$$P_{\Gamma M_{\Gamma}}^{\dagger} \equiv P_{J M_J T M_T}^{\dagger} \equiv (a_{\gamma}^{\dagger} \times a_{\gamma}^{\dagger})_{M_{\Gamma}}^{(\Gamma)} \equiv (a_{jt}^{\dagger} \times a_{jt}^{\dagger})_{M_J M_T}^{(JT)}, \quad (6)$$

where γ stands for jt , Γ for JT , m_{γ} for $m_j m_t$ and M_{Γ} for $M_J M_T$. An arbitrary pair state can then be written as

$$|\Gamma_1 \dots \Gamma_n \Lambda_2 \dots \Lambda_n\rangle \equiv \left(\dots \left(\left(P_{\Gamma_1}^{\dagger} \times P_{\Gamma_2}^{\dagger} \right)^{(\Lambda_2)} \times P_{\Gamma_3}^{\dagger} \right)^{(\Lambda_3)} \times \dots \times P_{\Gamma_n}^{\dagger} \right)^{(\Lambda_n)} |0\rangle, \quad (7)$$

which can be denoted in short as

$$|P_j^n\rangle \equiv |\Gamma_1 \dots \Gamma_n \Lambda_2 \dots \Lambda_n\rangle, \quad (8)$$

where the index j stands for the set $\{\Gamma_1 \dots \Gamma_n \Lambda_2 \dots \Lambda_{n-1}\}$, that is, the angular momenta and isospins Γ_q of the n pairs, and the intermediate angular momenta and isospins Λ_q . Note that Λ_1 (not shown) equals Γ_1 and that Λ_n is the total angular momentum and isospin, and therefore fixed and not included in j . Since Chen's algorithm³² is valid for arbitrary pairs, the analysis now proceeds as before, and consists of the construction of an orthonormal basis from the diagonalization

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of the overlap matrix $\langle P_j^n | P_{j'}^n \rangle$,

$$|\bar{P}_r^n\rangle = \sqrt{\frac{1}{O_r}} \sum_{j=1}^{\Omega} C_{rj} |P_j^n\rangle, \quad r = 1, \dots, \Omega, \quad (9)$$

where O_r and C_{rj} have the same meaning as in Eq. (4) but now in the full shell-model basis of dimension Ω . The diagonalization of the shell-model hamiltonian in that basis,

$$\langle \bar{P}_r^n | \hat{H}^f | \bar{P}_{r'}^n \rangle = \sqrt{\frac{1}{O_r O_{r'}}} \sum_{j,j'=1}^{\Omega} C_{rj} C_{r'j'} \langle P_j^n | \hat{H}^f | P_{j'}^n \rangle, \quad r, r' = 1, \dots, \Omega, \quad (10)$$

leads to the untruncated eigenspectrum of the shell model.

The B -pair content of an arbitrary shell-model state can now be analyzed as follows. First, a shell-model diagonalization is performed in a complete basis $|\bar{P}_r^n\rangle$ ($r = 1, \dots, \Omega$), leading to eigenstates

$$|\bar{E}_s^n\rangle = \sum_{r=1}^{\Omega} E_{sr} |\bar{P}_r^n\rangle, \quad s = 1, \dots, \Omega. \quad (11)$$

The B -pair content of a given eigenstate $|\bar{E}_s^n\rangle$ is the square of its projection onto the subspace spanned by B -pair states which equals

$$\langle \bar{E}_s^n | B^n \rangle^2 \equiv \sum_{k=1}^{\omega} |\langle \bar{E}_s^n | \bar{B}_k^n \rangle|^2, \quad (12)$$

where the overlap matrix element on the right-hand side can be expressed as

$$\langle \bar{E}_s^n | \bar{B}_k^n \rangle = \sum_{r,j=1}^{\Omega} \sum_{i=1}^{\omega} E_{sr} \sqrt{\frac{1}{O_r}} C_{rj} \sqrt{\frac{1}{o_k}} c_{ki} \langle P_j^n | B_i^n \rangle, \quad (13)$$

in terms of overlap matrix elements that can be computed with Chen's algorithm.³²

A final word is needed concerning the calculation of matrix elements of a shell-model hamiltonian between pair states as they appear on the right-hand sides of Eqs. (5) and (10). For the case of a single- j shell, the one-body part of \hat{H}^f gives rise to a constant and can be neglected. Its two-body part \hat{H}_2^f is entirely determined by the two-body matrix elements

$$v_{JT}^{2f} \equiv v_{JT}^{2f} \equiv \langle j^2 JT | \hat{H}_2^f | j^2 JT \rangle, \quad (14)$$

which enter as follows in the expression for the pair matrix element:

$$\begin{aligned} & \langle \Gamma_1 \dots \Gamma_n \Lambda_2 \dots \Lambda_n | \hat{H}_2^f | \Gamma'_1 \dots \Gamma'_n \Lambda'_2 \dots \Lambda'_n \rangle \\ &= \delta_{\Lambda_n \Lambda'_n} \left[\frac{4n - 2\gamma - 1}{2\gamma + 1} \langle \Gamma_1 \dots \Gamma_n \Lambda_2 \dots \Lambda_n | \Gamma'_1 \dots \Gamma'_n \Lambda'_2 \dots \Lambda'_n \rangle \sum_{\Gamma} (2\Gamma + 1) v_{\Gamma}^{2f} + \right. \\ & \quad \left. \sum_{\Gamma \Lambda} \frac{2\Lambda + 1}{2(2\Lambda_n + 1)} \langle \Gamma_1 \dots \Gamma_n \Gamma \Lambda_2 \dots \Lambda_n \Lambda | \Gamma'_1 \dots \Gamma'_n \Gamma \Lambda'_2 \dots \Lambda'_n \Lambda \rangle v_{\Gamma}^{2f} \right], \quad (15) \end{aligned}$$

with $2\gamma + 1 = 2(2j + 1)$, $2\Gamma + 1 = (2J + 1)(2T + 1)$, $2\Lambda_q + 1 = (2L_q + 1)(2T_q + 1)$, and so on. The second sum is over all possible pairs with angular momentum and isospin Γ which couples with Λ_n to all possible Λ , the total angular momentum and isospin of the $(n + 1)$ -pair state.

It is well known³⁶ that the limitation to a restricted model space (*e.g.*, a single- j shell) leads to an effective hamiltonian with higher-order interactions (see Refs. 37 and 38 for a recent discussion of $T = 1$ three-body interactions in the $1f_{7/2}$ shell). Equations (5) and (10) are generally valid, irrespective of the order of the interaction in \hat{H}^f . Equation (15), on the other hand, is specific to a two-body interaction but it can be readily generalized to higher orders. The corresponding expression for a three-body interaction, for example, involves the same overlap matrix elements as those in Eq. (15) with in addition overlaps between states of $(n + 1)$ pairs plus one particle. These can be computed with the NPSM algorithm generalized to odd-mass nuclei.³⁹

In the present review the order of the interactions in the shell-model hamiltonian is limited to two-body and lowest-order transition operators are taken.

5. Boson mapping

The boson equivalent of the basis (1) is

$$|b^n L_2 \dots L_n\rangle \equiv \left(\dots \left((b^\dagger \times b^\dagger)^{(L_2)} \times b^\dagger \right)^{(L_3)} \times \dots \times b^\dagger \right)^{(L_n)} |0\rangle, \quad (16)$$

where b^\dagger creates a boson with angular momentum (or spin) $\ell = 2j$ and isospin $t = 0$. While the angular momentum coupling is the same in Eqs. (1) and (16), overlap and hamiltonian matrix elements are different in both bases because of the internal structure of the pairs, in contrast to the assumed elementary character of the bosons. Nevertheless, Pauli corrections can be systematically applied to the boson calculation in the following way. In general, for $n > 2$, the boson basis (16) is non-orthogonal and overcomplete. As in the fermion case, the diagonalization of the overlap matrix

$$\langle b_i^n | b_j^n \rangle \equiv \langle b^n L_2 \dots L_{n-1} L_n | b^n L'_2 \dots L'_{n-1} L_n \rangle, \quad (17)$$

leads to an orthogonal basis of linearly independent vectors. For technical reasons that have to do with the computation of coefficients of fractional parentage (CFPs), it is in this case more convenient to define an orthonormal basis via a Gram-Schmidt procedure. For a given sequence of linearly independent, non-orthogonal n -boson

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states $|b_i^n\rangle$ ($i = 1, \dots, \omega'$), an orthogonal series can be defined as follows:

$$\begin{aligned} |\tilde{b}_1^n\rangle &= \langle b_1^n | b_1^n \rangle^{-1/2} |b_1^n\rangle, \\ |\tilde{b}_2^n\rangle &= (\langle b_2^n | b_2^n \rangle - \langle b_2^n | \tilde{b}_1^n \rangle^2)^{-1/2} (|b_2^n\rangle - \langle b_2^n | \tilde{b}_1^n \rangle |\tilde{b}_1^n\rangle), \\ &\vdots \\ |\tilde{b}_i^n\rangle &= \left(\langle b_i^n | b_i^n \rangle - \sum_{j=1}^{i-1} \langle b_i^n | \tilde{b}_j^n \rangle^2 \right)^{-1/2} \left(|b_i^n\rangle - \sum_{j=1}^{i-1} \langle b_i^n | \tilde{b}_j^n \rangle |\tilde{b}_j^n\rangle \right), \end{aligned} \quad (18)$$

until $i = \omega'$. To establish the connection with the orthogonal fermion-pair series, an additional transformation is needed,

$$|\bar{b}_k^n\rangle = \sum_{i=1}^{\omega} c_{ki} |\tilde{b}_i^n\rangle, \quad k = 1, \dots, \omega, \quad (19)$$

in terms of the coefficients c_{ki} defined in Eq. (4). Because of the orthogonality of the basis $|\tilde{b}_i^n\rangle$ and the properties of the coefficients c_{ki} , the basis $|\bar{b}_k^n\rangle$ is orthogonal and is the boson equivalent of the fermion basis $|\bar{B}_k^n\rangle$. The matrix elements of the boson hamiltonian in this basis are therefore determined from

$$\langle \bar{b}_k^n | \hat{H}^b | \bar{b}_{k'}^n \rangle = \langle \bar{B}_k^n | \hat{H}^f | \bar{B}_{k'}^n \rangle, \quad k, k' = 1, \dots, \omega. \quad (20)$$

With use of the inverse of the relation (19), of the equality (20) and of Eq. (5), the matrix elements of the boson hamiltonian in the orthogonal basis $|\tilde{b}_i^n\rangle$ can be written in terms of those of the shell-model hamiltonian in the fermion-pair basis,

$$\langle \tilde{b}_i^n | \hat{H}^b | \tilde{b}_{i'}^n \rangle = \sum_{k, k'=1}^{\omega} \sqrt{\frac{1}{O_k O_{k'}}} c_{ki} c_{k'i'} \sum_{j, j'=1}^{\omega} c_{kj} c_{k'j'} \langle B_j^n | \hat{H}^f | B_{j'}^n \rangle, \quad i, i' = 1, \dots, \omega. \quad (21)$$

Three additional technical issues must be ironed out. First, for a given total angular momentum L_n , the number of linearly independent boson states (16) may be larger than the corresponding number of fermion-pair states (1), $\omega \leq \omega'$, indicating that there are $\omega' - \omega$ spurious boson states which are Pauli forbidden in the fermion space. The matrix elements of the boson hamiltonian pertaining to these states remain undefined in Eq. (21). Since these states are spurious, they must be eliminated from the boson space, implying the following choice of boson matrix elements:

$$\begin{aligned} \langle \tilde{b}_i^n | \hat{H}^b | \tilde{b}_i^n \rangle &= +\infty, \quad i = \omega + 1, \dots, \omega', \\ \langle \tilde{b}_i^n | \hat{H}^b | \tilde{b}_{i'}^n \rangle &= \langle \tilde{b}_{i'}^n | \hat{H}^b | \tilde{b}_i^n \rangle = 0, \quad i \leq \omega < i' \leq \omega'. \end{aligned} \quad (22)$$

The second technical issue concerns the fact that Eq. (21) defines the entire boson hamiltonian up to and including n -body interactions. To isolate its n -body part \hat{H}_n^b , one should subtract the previously determined n' -body interactions, $n' < n$. The procedure is straightforward but rather cumbersome to write down explicitly

up to all orders. Up to the three-body interactions that will be considered below, one has the following results. The single-boson energy is determined from

$$\epsilon_b \equiv \langle b | \hat{H}^b | b \rangle = \langle B | \hat{H}^f | B \rangle, \quad (23)$$

which is nothing but the shell-model matrix element v_{JT}^{2f} in the aligned neutron–proton configuration with $J = 2j$ and $T = 0$. The two-body part of the boson hamiltonian is determined from

$$v_{L_2}^{2b} \equiv \langle b^2 L_2 | \hat{H}_2^b | b^2 L_2 \rangle = \langle b^2 L_2 | \hat{H}^b | b^2 L_2 \rangle - 2\epsilon_b, \quad (24)$$

where it is assumed that the two-boson states are normalized such that the matrix element of the total boson hamiltonian can be taken from Eq. (21). The three-body part of the boson hamiltonian follows from

$$\begin{aligned} \langle b^3 [\tilde{L}_2] L_3 | \hat{H}_3^b | b^3 [\tilde{L}'_2] L_3 \rangle &= \langle b^3 [\tilde{L}_2] L_3 | \hat{H}^b | b^3 [\tilde{L}'_2] L_3 \rangle - 3\epsilon_b \delta_{\tilde{L}_2 \tilde{L}'_2} - \\ & 3 \sum_{L'_2} [\ell^2 [L'_2] \ell] \ell^3 [\tilde{L}_2] L_3 [\ell^2 [L'_2] \ell] \ell^3 [\tilde{L}'_2] L_3 v_{L'_2}^{2b}, \end{aligned} \quad (25)$$

where again the matrix element of the total boson hamiltonian on the right-hand side are calculated from Eq. (21). Equation (25) requires some explanation. The basis consisting of the three-boson states

$$|b^3 L_2 L_3 \rangle \equiv \left((b^\dagger \times b^\dagger)^{(L_2)} \times b^\dagger \right)^{(L_3)} |o\rangle, \quad (26)$$

is non-orthogonal and non-normalized. The intermediate angular momentum L_2 can be used as a label and, after the application of Eq. (18), one arrives at an orthogonal basis denoted by $|b^3 [\tilde{L}_2] L_3 \rangle$, with the notation $[\tilde{L}_2]$ as a reminder of the Gram–Schmidt procedure. This basis can be used to express the matrix elements of a two-body interaction in the usual manner with CFPs.⁴⁰ In general, the matrix element of an n' -body boson hamiltonian between n -boson states ($n \geq n'$) can be written as

$$\langle \tilde{b}_i^n | \hat{H}_{n'}^b | \tilde{b}_{i'}^{n'} \rangle = \frac{n!}{n!(n-n')!} \sum_{jkk'} [\tilde{b}_j^{n-n'} \tilde{b}_k^{n'} | \tilde{b}_i^n] [\tilde{b}_j^{n-n'} \tilde{b}_k^{n'} | \tilde{b}_{i'}^{n'}] \langle \tilde{b}_k^{n'} | \hat{H}_{n'}^b | \tilde{b}_k^{n'} \rangle, \quad (27)$$

in terms of $n \rightarrow n - n'$ CFPs. The third term on the right-hand side of Eq. (25) arises from the application of this result for $n = 3$ and $n' = 2$, together with the explicit notation of CFPs for bosons with spin ℓ .

The third technical issue concerns the hierarchy of states since, in general, the definition of the interactions between the bosons depends on the order of states chosen in the Gram–Schmidt procedure (19). In the mapping from B pairs to b bosons no ambiguity exists for the two-body interaction between the bosons ($n = 2$) since states are unique for a given angular momentum J . This is no longer the case for $n \geq 3$ and as a result there exist many different n -body interactions that exactly reproduce the shell-model results in the B^n space. The method followed here is to define a hierarchy based on the importance of the overlap with the yrast shell-model

state (see Sect. 7 for examples), leading to a procedure which, for $n = 3$, can be summarized in the following steps.

- Construct and diagonalize the shell-model hamiltonian in the B^3 basis, leading to the eigenvalues E_k ($k = 1, \dots, \omega$).
- To deal with spurious states, the set of ω eigenvalues is enlarged with $E_k = +\infty$ ($k = \omega' - \omega + 1, \dots, \omega'$) (*i.e.*, $\omega' - \omega$ large values in numerical applications).
- Construct and diagonalize the boson hamiltonian with up to two-body interactions in the analogue b^3 basis. This is achieved by computing the second and third terms on the left-hand side of Eq. (25) which after diagonalization yields the eigenvalues E'_k ($k = 1, \dots, \omega'$) with corresponding eigenvectors c'_{kl} ($l = 1, \dots, \omega'$).
- The three-body interaction in the analogue boson basis is obtained by transforming back the matrix with the differences $E_k - E'_k$ on the diagonal,

$$v_{ll'}^{3b} = \sum_{k=1}^{\omega'} c'_{lk} (E_k - E'_k) c'_{kl'}, \quad (28)$$

where l and l' are short-hand notations for the three-boson labels $[\tilde{L}_2]L_3$ and $[\tilde{L}'_2]L_3$.

Since the three-body components of the boson interaction are found to be small (see Sect. 7), no exhaustive study of the three-body aspect of the mapping is attempted in this review.

6. Three approximations

Let us now take stock of the situation with regard to the aligned-pair approximation as described in the technical discussion of the previous two sections. A first possibility is to determine the $T = 0$ spectrum of a $2n$ -particle system by diagonalizing a given shell-model hamiltonian \hat{H}^f in the aligned-pair basis $|\tilde{B}_k^n\rangle$. This is a truncated shell-model calculation in which the Pauli principle is fully taken into account and no boson mapping is needed. The calculation becomes more difficult as n increases because of the complexity of Chen's algorithm. This truncated shell-model calculation can be replaced *exactly* by its boson equivalent *if* the mapped boson hamiltonian \hat{H}^b is determined up to all orders (*i.e.*, up to order n for a $2n$ -particle system). The Pauli principle is obeyed by means of appropriate interactions between the bosons. No simplification of the original problem is obtained since the determination of \hat{H}^b up to order n requires the calculation of matrix elements of \hat{H}^f in the aligned-pair basis $|\tilde{B}_k^n\rangle$ [see Eq. (21)]. Significant simplifications may result, however, if the mapped boson hamiltonian \hat{H}^b is determined up to an order $n' < n$ but this simplification is at the expense of some violation of the Pauli principle.

In Sect. 7 the above statements are illustrated with examples. Since a number of approximations are made at different stages, it is useful to enounce these approx-

imations and to indicate whether they can be tested theoretically and/or experimentally. Let us start from the general observation that the $N = Z$ nuclei under consideration can be described in the spherical shell model if a sufficiently large model space with an appropriate interaction is adopted. With this as a premise the following assumptions are made to arrive at an approximation in terms of aligned-pair bosons.

- (i) *The shell-model space is truncated to a single high- j orbit.* A theoretical test of this assumption is not attempted in this review. Its validity clearly depends on the specific features of the initial shell-model hamiltonian. Two particular mass regions where the approximation might be valid spring to mind: $N = Z$ nuclei in the $1f_{7/2}$ and $1g_{9/2}$ shells. More important is that the assumption can be tested experimentally, as illustrated with examples in Sect. 7.
- (ii) *The single- j shell space is reduced to one written in terms of aligned B pairs.* Some dependence exists on the shell-model interaction adopted in the single- j shell space. Nevertheless, if a reasonable interaction is taken, this assumption can be tested generically. Furthermore, the extension of the NPSM that includes isospin³⁴ is the appropriate formalism to test the combined approximations (i) and (ii). A recent calculation of this type,³⁵ which starts from a realistic shell-model space and interaction, seem to indicate that the combined approximations (i) and (ii) hold fairly well in $N = Z$ nuclei from ^{92}Pd to ^{100}Sn .
- (iii) *The aligned B pairs are replaced by b bosons.* As argued in the previous section, if the boson hamiltonian is calculated up to all orders, the mapping is exact and no approximation is made. The usual procedure, however, is to map up to two-body boson interactions which implies some amount of Pauli violation. In the next section the validity of the two-body boson mapping is tested by calculating the effect of the three-body interaction.

7. Applications

A number of results can be established for a shell with arbitrary j . They are useful in the discussion of specific cases, in particular the $1f_{7/2}$ and $1g_{9/2}$ shells.

7.1. Any j shell

The M1 operator in the shell model is given by

$$\hat{T}_{\mu}^{\text{f}}(\text{M1}) = \sqrt{\frac{3}{4\pi}} \left(\sum_{i \in \nu} g_{\nu}^{\text{s}} s_{\mu}(i) + \sum_{i \in \pi} [g_{\pi}^{\ell} \ell_{\mu}(i) + g_{\pi}^{\text{s}} s_{\mu}(i)] \right), \quad (29)$$

where the sums are over neutrons and protons, and in each sum appear the orbital and spin gyromagnetic factors, g_{ρ}^{ℓ} and g_{ρ}^{s} , with $\rho = \nu$ for a neutron and $\rho = \pi$ for a proton. For the calculation of magnetic moments (*i.e.*, diagonal matrix elements) the M1 operator (29) can be replaced by one in terms of neutron and

proton g factors. In second quantization the z component of the latter operator can be written as

$$\hat{\mu}_0^f = \sqrt{\frac{j(j+1)(2j+1)}{3}} \left[g_\nu (\nu_j^\dagger \times \tilde{\nu}_j)_0^{(1)} + g_\pi (\pi_j^\dagger \times \tilde{\pi}_j)_0^{(1)} \right], \quad (30)$$

where ρ_{jm}^\dagger creates a neutron ($\rho = \nu$) or a proton ($\rho = \pi$) in the j shell, and $\tilde{\rho}_{jm} = (-)^{j+m} \rho_{j-m}$. This operator can be written alternatively as a sum of an isoscalar part, multiplied by $(g_\nu + g_\pi)$, and an isovector part, multiplied by $(g_\nu - g_\pi)$. For the M1 matrix elements between states in a single- j shell of the same isospin T and with projection $T_z = 0$, only the former part contributes and, since the isoscalar part is proportional to the angular momentum operator, it follows that the g factor of any state in an $N = Z$ nucleus equals $(g_\nu + g_\pi)/2$. This result is generally valid under the assumptions that isospin is a good quantum number and that the nucleons are confined to a single- j shell.⁴¹

In terms of b bosons the M1 operator is of the form

$$\hat{T}_\mu^b(\text{M1}) = \sqrt{\frac{3}{4\pi}} g_b \hat{J}_\mu = \sqrt{\frac{2j(2j+1)(4j+1)}{4\pi}} g_b (b^\dagger \times \tilde{b})_\mu^{(1)}. \quad (31)$$

The g factor of the b boson, g_b , is obtained from the g factor of the B pair which, due to the above argument, equals $(g_\nu + g_\pi)/2$. Since the operator (31) is proportional to the angular momentum operator, one finds that the g factor of any state $|\alpha J\rangle$ in the boson model equals

$$g(\alpha J) \equiv \frac{\mu(\alpha J)}{J} = \sqrt{\frac{4\pi}{3}} \frac{\langle \alpha J J | \hat{T}_0^b(\text{M1}) | \alpha J J \rangle}{J} = g_b = \frac{g_\nu + g_\pi}{2}. \quad (32)$$

One recovers therefore the shell-model result that the g factor of any state in an $N = Z$ nucleus equals $(g_\nu + g_\pi)/2$.

The conclusion of the preceding discussion is that magnetic moments do not provide a test of the assumptions (ii) and (iii) of Sect. 6 since any $T = T_z = 0$ state in a single- j shell has a g factor equal to $(g_\nu + g_\pi)/2$, irrespective of whether this state can be written in terms of B pairs or not, and since the same result is obtained with b bosons. However, deviations from $(g_\nu + g_\pi)/2$ are indicative of admixtures of configurations beyond a single- j shell and therefore magnetic moments constitute a test of assumption (i).

The E2 operator in the shell model is

$$\hat{T}_\mu^f(\text{E2}) = e_\nu \sum_{i \in \nu} r_i^2 Y_{2\mu}(\theta_i, \phi_i) + e_\pi \sum_{i \in \pi} r_i^2 Y_{2\mu}(\theta_i, \phi_i), \quad (33)$$

where each sum is multiplied with the appropriate effective charge. In a single- j shell the second-quantized form of this operator is

$$\hat{T}_\mu^f(\text{E2}) = -x_j \left(N + \frac{3}{2} \right) l_{\text{ho}}^2 \left[e_\nu (\nu_j^\dagger \times \tilde{\nu}_j)_\mu^{(2)} + e_\pi (\pi_j^\dagger \times \tilde{\pi}_j)_\mu^{(2)} \right], \quad (34)$$

Table 1. Shell-model matrix elements v_{JT}^{2f} (in MeV) in the $1f_{7/2}$ and $1g_{9/2}$ shells.

(JT)	(01)	(10)	(21)	(30)	(41)	(50)	(61)	(70)	(81)	(90)
^{42}Sc	-3.187	-2.576	-1.601	-1.697	-0.372	-1.677	0.055	-2.571		
^{54}Co	-2.551	-1.614	-1.105	-0.730	0.101	-0.664	0.349	-2.354		
SLGT0	-2.392	-1.546	-0.906	-0.747	-0.106	-0.423	0.190	-0.648	0.321	-1.504

where N is the major oscillator quantum number [$N = 2(n - 1) + \ell$] and l_{ho} is the length parameter of the harmonic oscillator, and with

$$x_j = \left[\frac{(2j - 1)(2j + 1)(2j + 3)}{64\pi j(j + 1)} \right]^{1/2}. \quad (35)$$

In terms of b bosons the E2 operator is of the form

$$\hat{T}_\mu^{\text{b}}(\text{E2}) = e_b(b^\dagger \times \tilde{b})_\mu^{(2)}, \quad (36)$$

where the boson effective charge e_b is obtained from the condition

$$\langle B || \hat{T}^{\text{f}}(\text{E2}) || B \rangle = \langle b || \hat{T}^{\text{b}}(\text{E2}) || b \rangle. \quad (37)$$

The two-particle matrix element on the left-hand side of Eq. (37) can be readily derived with the help of Eq. (34), leading to the result

$$e_b = -(e_\nu + e_\pi) \left(N + \frac{3}{2}\right) l_{\text{ho}}^2 \left[\frac{(2j - 1)^2(2j + 1)(4j + 1)(4j + 3)}{128\pi j(j + 1)^2(4j - 1)} \right]^{1/2}. \quad (38)$$

Unlike the case of M1 properties, no general conclusions can be drawn for E2 transitions and moments. The preceding expressions are nevertheless helpful in the discussion of nuclei in the two shells of interest.

7.2. The $1f_{7/2}$ shell

The restriction to a single- j shell is an approximation which, if valid at all, induces higher-order interactions in the effective shell-model hamiltonian that should be calculated from perturbation theory.³⁶ To avoid the complexities associated with three- and higher-body interactions between nucleons, a more phenomenological approach is followed here which consists of introducing a two-body interaction that depends on the mass number A . The spectra of the nuclei ^{42}Sc and ^{54}Co are well known⁴² and allow the determination of the particle-particle and hole-hole matrix elements, respectively, up to a constant. This constant is determined from measured binding energies⁴³ of neighbouring nuclei, leading to the shell-model matrix elements shown in Table 1. The interaction appropriate for $N = Z$ nuclei intermediate between ^{42}Sc and ^{54}Co is obtained from linear interpolation,

$$v_{JT}^{2f}(A) = \frac{54 - A}{12} v_{JT}^{2f}(^{42}\text{Sc}) + \frac{A - 42}{12} v_{JT}^{2f}(^{54}\text{Co}). \quad (39)$$

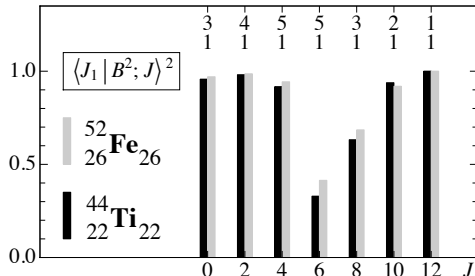


Fig. 3. Overlaps of the yrast eigenstates in the $(1f_{7/2})^4$ system, for angular momentum J and isospin $T = 0$, with the B -pair state $|B^2; J\rangle$. The shell-model interaction is defined in Eq. (39). Also shown are the numbers of $(1f_{7/2})^4$ states (top) and of B -pair states (bottom) with angular momentum J and isospin $T = 0$.

The advantage of using a two-body interaction is that particle-hole symmetry is preserved. The calculation of a nucleus heavier than ^{48}Cr , corresponding to the space $(1f_{7/2})^{2n}$ with $n > 4$, can be replaced by one in the space $(1f_{7/2})^{16-2n}$. All properties are identical except quadrupole moments which change sign.⁴⁴ This simplifies the calculation in the pair basis $|P_j^n\rangle$ which for $n > 4$ can be replaced by $|P_j^{8-n}\rangle$.

7.2.1. ^{44}Ti and ^{52}Fe

For two neutrons and two protons (both particle- or hole-like) the B -pair state is unique for a given total angular momentum J and isospin $T = 0$. The B -pair content of a given shell-model state can be obtained from Eq. (12) with $\omega = 1$. This quantity is shown in Fig. 3 for the yrast states in ^{44}Ti and ^{52}Fe . Most yrast states have a large B -pair content but not for $J = 6$ and $J = 8$. It seems as if the two B pairs do not like to couple to a total angular momentum which equals their individual spins. Although the interaction varies considerably with mass (see Table 1), similar results are found in ^{44}Ti and ^{52}Fe , indicating that these conclusions are robust as long as a reasonable nuclear interaction is used.

Also shown in Fig. 3 are the numbers of $(1f_{7/2})^4$ states and of B -pair states with angular momentum J and isospin $T = 0$. This allows one to judge whether the observation of a high overlap is trivial or meaningful. For example, only one shell-model state exists with $J = 12$ and $T = 0$ which therefore necessarily has an overlap of 1 with the B -pair state. In contrast, four shell-model states exist with $J = 2$ and $T = 0$ but it is found that the yrast eigenstate has an overlap of more than 0.98 with a single B -pair state. The latter is a physically meaningful result whereas the former is trivial.

The energy spectra of ^{44}Ti and ^{52}Fe , shown in Figs. 4 and 5, confirm the above wave-function analysis. For the sake of comparison with the data, the shell-model energy of the 0^+ level is normalized to zero, and it is seen that the excitation spec-

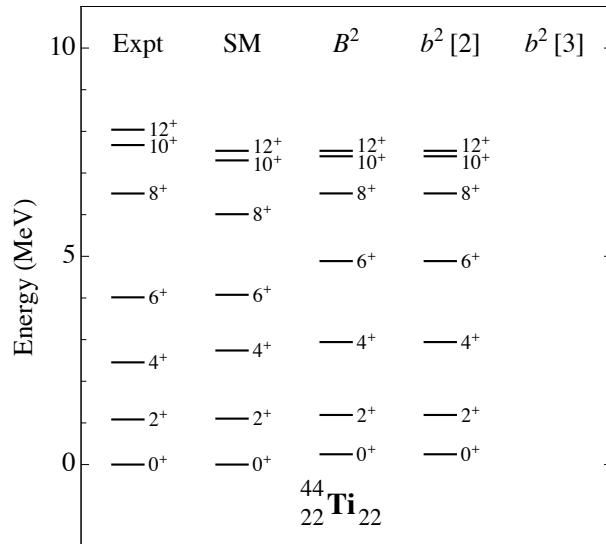


Fig. 4. The yrast spectrum of ^{44}Ti . Levels are labelled by their angular momentum and parity J^π . The different columns contain the experimental⁴² levels (Expt), the results of the $(1f_{7/2})^4$ shell model (SM) with the interaction (39), the expectation value of the shell-model hamiltonian in the B -pair state (B^2) and the expectation value of the mapped boson hamiltonian with up to two-body interactions (b^2 [2]). The shell-model energy of the 0_1^+ level is normalized to zero.

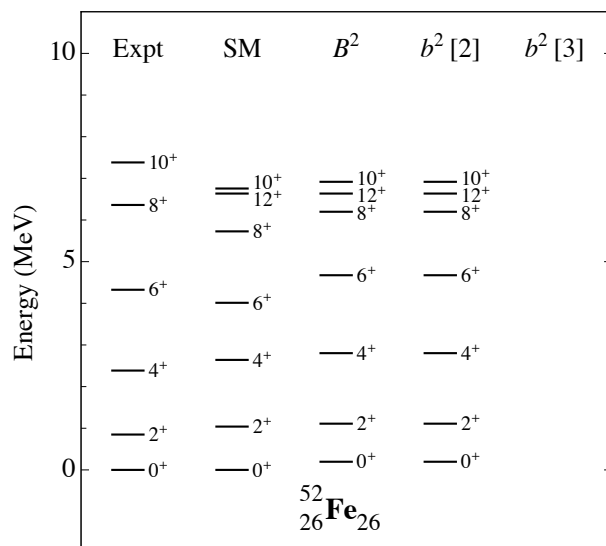


Fig. 5. Same as Fig. 4 for ^{52}Fe .

Table 2. Coefficients $a_{JT}^L(j)$ in the expansion (40) for $j = 7/2$.

(JT)	L	0	2	4	6	8	10	12
(01)		19305	1287	3315	14535			
		13732	1312	8612	245288			
(10)		35035	2821	11305	110789	38		
		41196	3936	25836	735864	2913		
(21)		21021	2379	16711	332367	399		
		13732	1312	8612	245288	971		
(30)		22295	3969	145775	1779141	5047	115	
		151052	14432	284196	2698168	10681	1221	
(41)		9555	2949	62475	4007955	45465	2415	
		151052	14432	94732	2698168	21362	1628	
(50)		245	147	16415	46305	12515	4711	6
		178516	17056	335868	245288	25246	5772	13
(61)		15	21	1435	270627	9843	2469	3
		151052	14432	94732	2698168	21362	1628	
(70)		1	197	2293	10337	15589	1897	7
		1716	-562848	3694548	8094504	833118	21164	13

 Table 3. Boson interaction matrix elements v_L^{2b} (in MeV) appropriate for the $1f_{7/2}$ and $1g_{9/2}$ shells.

L	0	2	4	6	8	10	12	14	16	18
^{42}Sc	-3.187	-2.576	-1.601	-1.697	-0.372	-1.677	0.055	$+\infty$		
^{54}Co	-2.551	-1.614	-1.105	-0.730	0.101	-0.664	0.349	$+\infty$		
SLGT0	-5.635	-4.956	-3.694	-2.333	-1.209	-0.455	-0.062	0.058	-0.506	$+\infty$

tra calculated in the shell model (SM) are reasonably close to the observed ones. The column ‘ B^2 ’ shows the expectation value of the shell-model hamiltonian in the B -pair state $|B^2; J, T = 0\rangle$. Note that absolute energies are calculated which are plotted relative to the shell-model 0_1^+ level. Therefore, the differences in energy between corresponding levels in the ‘SM’ and ‘ B^2 ’ columns correlate with the overlaps shown in Fig. 3. For example, the difference is greatest for $J^\pi = 6^+$ since for this state the overlap is smallest.

The two-boson calculation with up to two-body interactions, shown in the column ‘ $b^2[2]$ ’ of Fig. 3, reproduces exactly the B -pair calculation, in agreement with the discussion of Sect. 5. Since, for a given angular momentum J and isospin $T = 0$, the mapping from two B pairs to two b bosons is one-to-one, simple expressions are found for the boson interaction matrix elements v_L^{2b} in terms of the two-body fermion matrix elements v_{JT}^{2f} . These relations are of the generic form

$$v_L^{2b} = \sum_{JT} a_{JT}^L(j) v_{JT}^{2f}, \quad (40)$$

with coefficients $a_{JT}^L(j)$ that depend on the single-particle angular momentum j of the shell. The coefficients for $j = 7/2$ are given in Table 2 and the resulting boson interaction matrix elements v_L^{2b} in Table 3. There is no four-particle shell-model state with $J = 14$, implying the choice $v_{14}^{2b} = +\infty$, in line with the recipe (22). In numerical calculations a large repulsive matrix element is taken.

No three-body interactions between the bosons intervene in ^{44}Ti and ^{52}Fe .

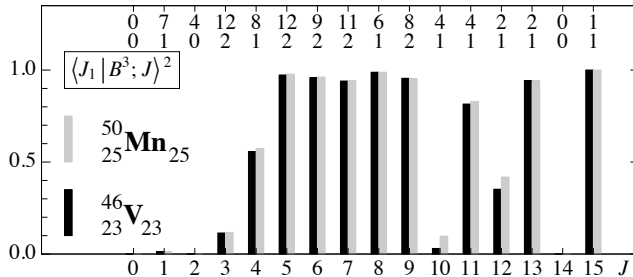


Fig. 6. The square of the projection of the yrast eigenstates in the $(1f_{7/2})^6$ system onto the subspace spanned by the B -pair states $|B^3; J\rangle$, for angular momentum J and isospin $T = 0$. The shell-model interaction is defined in Eq. (39). Also shown are the numbers of $(1f_{7/2})^6$ states (top) and of B -pair states (bottom) with angular momentum J and isospin $T = 0$.

7.2.2. ^{46}V and ^{50}Mn

Odd-odd $N = Z$ nuclei are of particular interest with regard to the question of the relevance of neutron-proton pairs. For three neutrons and three protons (both particle- or hole-like) in a $j = 7/2$ shell, there are at most two linearly independent B -pair states for a given total angular momentum J and isospin $T = 0$. The B -pair content of a shell-model state can therefore be obtained from Eq. (12) with $\omega = 1$ or 2. This quantity is shown in Fig. 6 for yrast states in ^{46}V and ^{50}Mn . On top of the figure are shown the numbers of $(1f_{7/2})^6$ states and of B -pair states with angular momentum J and isospin $T = 0$, in order to judge whether a large overlap is a physically meaningful or a trivial result.

A surprising feature of the results of Fig. 6 is the ‘schizophrenic’ behaviour of $T = 0$ states in ^{46}V and ^{50}Mn , with most having either a large or a small B -pair component. Clearly, only the former states can be interpreted in terms of B pairs or b bosons, as will be shown below. Before doing so, a few words are in order about those states that do *not* conform to such a description. The most obvious example is the $J = 2$ state which simply cannot be constructed out of three B pairs. A wave-function analysis with the method outlined in Sect. 4, gives $|SPD; 2\rangle$ as its main component, $\langle 2_1^+ | SPD; 2 \rangle^2 = 0.825$ (0.841) in ^{46}V (^{50}Mn), where S , P and D are pairs with $J = 0, T = 1$, $J = 1, T = 0$ and $J = 2, T = 1$, respectively. All low-spin states can in fact be adequately written in terms of the S , P , D and F pairs that correspond to the bosons of IBM-4, confirming the analysis of Juillet *et al.*²³ in a different mass region. The most remarkable state of this kind is the yrast 1^+ level which approximately can be written as $|P^3; 1\rangle$ since $\langle 1_1^+ | P^3; 1 \rangle^2 = 0.725$ (0.728) in ^{46}V (^{50}Mn). (Note that there is only one P^3 state with angular momentum $J = 1$ since $|P^3[0]1\rangle \propto |P^3[2]1\rangle$.)

The results of the wave-function analysis are confirmed by the energy spectra shown in Figs. 7 and 8. The observed spectra are reasonably well reproduced in the shell-model calculation, the main deficiency of the latter being that it cannot

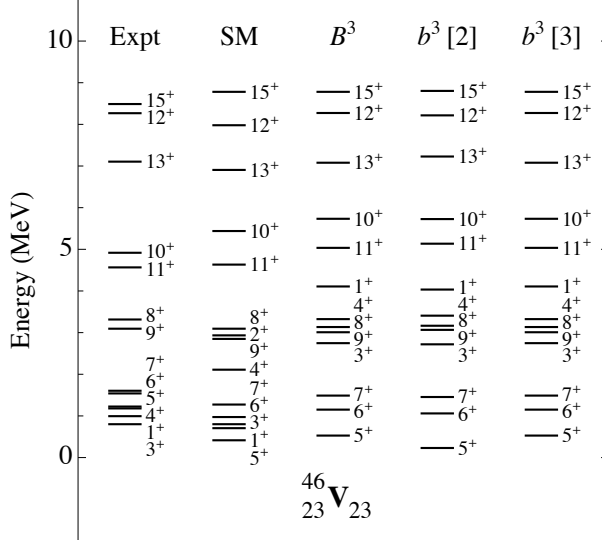


Fig. 7. The spectrum of yrast states in ${}^{46}\text{V}$ with isospin $T = 0$. Levels are labelled by their angular momentum and parity J^π . The different columns contain the experimental⁴² levels (Expt), the results of the $(1f_{7/2})^6$ shell model (SM) with the interaction (39), the lowest eigenvalue of the shell-model hamiltonian in the B -pair subspace (B^3) and the lowest eigenvalue of the mapped boson hamiltonian with up to two-body ($b^3[2]$) and up to three-body ($b^3[3]$) interactions. The shell-model energy of the $T = 0$ ground state, $J^\pi = 3^+$, is normalized to the experimental excitation energy of this level which is relative to the 0^+ ground state with isospin $T = 1$.

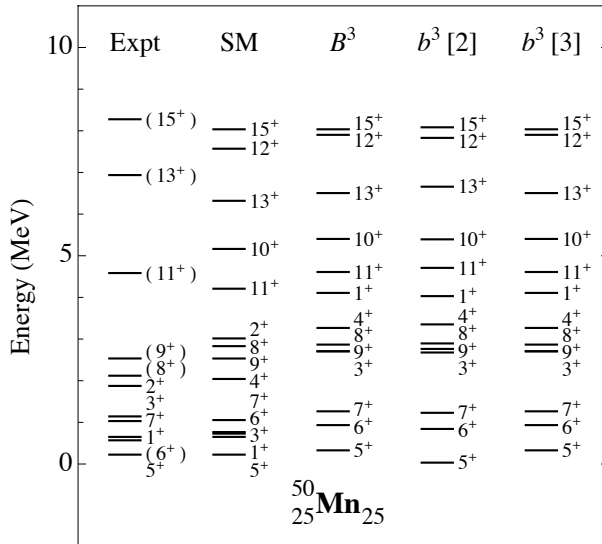


Fig. 8. Same as Fig. 7 for ${}^{50}\text{Mn}$. The shell-model energy of the $T = 0$ ground state, $J^\pi = 5^+$, is normalized to the experimental excitation energy of this level which is relative to the 0^+ ground state with isospin $T = 1$.

account for the 3^+-5^+ inversion of $T = 0$ ground states between ^{46}V and ^{50}Mn . The same level of agreement is found in the B -pair calculation except that the low-spin states (1^+ , 2^+ , 3^+ and 4^+) are at much higher energies (or absent in the case of the 2^+ level), in disagreement with the data.

The mapped two-body boson hamiltonian (column ‘ $b^3[2]'$ ’) closely reproduces the B^3 calculation, including its deficient low-spin levels. Consequently, the three-body components of the interaction between the b bosons are small. Let us consider two examples to illustrate the calculation of three-body interactions between the bosons, namely those pertaining to the 5^+ and 7^+ states. Numerical values are quoted for ^{46}V , the results obtained for ^{50}Mn being similar. For $J = 5$ there are two independent fermionic B^3 states and the diagonalization of the shell-model hamiltonian in this basis yields the eigenvalues $\{E_k\} = \{-20.599, -18.456\}$, in MeV. The same number of independent bosonic b^3 states exists, which can be chosen as $|b^3[\tilde{L}_2]5\rangle$ with $\tilde{L}_2 = 12$ and 2. The first state in this basis is taken as $\tilde{L}_2 = 12$ because its fermionic analogue, $|B^3[1\tilde{2}]5\rangle$, has maximum overlap with the shell-model 5_1^+ state. The second state in the boson basis is orthogonal to $|b^3[1\tilde{2}]5\rangle$ and therefore unique, and hence its \tilde{L}_2 can be chosen freely. The diagonalization of the mapped one-plus-two-body boson hamiltonian in this basis leads to the eigenvalues $\{E'_k\} = \{-20.897, -18.391\}$, in MeV. The transformation (28) of the differences $\{E_k - E'_k\} = \{0.298, -0.065\}$ back to the orthogonal boson basis leads to the three-body interaction (in MeV)

$$\langle b^3[\tilde{L}_2]5|\hat{H}_3^b|b^3[\tilde{L}'_2]5\rangle = \begin{bmatrix} 0.290 & 0.053 \\ 0.053 & -0.057 \end{bmatrix}, \quad L_2, L'_2 = 12, 2. \quad (41)$$

The $J = 7$ interaction can be dealt with in a similar way. There are two independent fermionic B^3 states and the diagonalization of the shell-model hamiltonian in the B -pair space leads to the eigenvalues $\{E_k\} = \{-19.639, -16.905\}$, in MeV. In this case there are *three* independent bosonic states $|b^3[\tilde{L}_2]7\rangle$ and the choice $\tilde{L}_2 = 0, 12$ and 2 maximizes the overlap with the shell-model 7_1^+ state. The diagonalization of the one-plus-two-body boson hamiltonian in this basis yields the eigenvalues $\{E'_k\} = \{-19.673, -16.925, +\infty\}$, in MeV. The spurious state in the three-boson system is thus removed by the two-body interaction matrix element $v_{14}^{2b} = +\infty$. Nevertheless, the entire 3×3 matrix must be used to define the three-body interaction for $J = 7$. This is achieved by transforming the differences $\{E_k - E'_k\} = \{0.034, -0.020, 0.000\}$ back to the orthogonal boson basis, leading to the three-body interaction (in MeV)

$$\langle b^3[\tilde{L}_2]7|\hat{H}_3^b|b^3[\tilde{L}'_2]7\rangle = \begin{bmatrix} 0.019 & 0.014 & -0.010 \\ 0.014 & 0.020 & 0.003 \\ -0.010 & 0.003 & 0.017 \end{bmatrix}, \quad L_2, L'_2 = 0, 12, 2. \quad (42)$$

Typically, the three-body matrix elements are of the order of a few tens of keV, the matrix element $\langle b^3[1\tilde{2}]5|\hat{H}_3^b|b^3[1\tilde{2}]5\rangle$ in Eq. (41) being by far the largest three-body correction in the $1f_{7/2}$ shell.

It will not have escaped the attention of the diligent reader that the dimensions of all hamiltonian matrices in the different approximations are small. The largest dimension, in the shell-model calculation for six nucleons with $J = 3$ or 5 and $T = 0$, is twelve. Modern shell-model codes usually adopt an m -scheme basis without good angular momentum and isospin but, even so, dimensions in a single- j shell do remain modest, of the order of a few hundred at most. Why then, this diligent reader might well ask, bother to seek a further reduction of dimension in terms of B pairs which introduces major computational complications? The answer is that conceptual insight is gained.

Let us illustrate this with the example of the yrast 5^+ state in ^{46}V or ^{50}Mn . According to Fig. 6 this state has a large component in the B -pair space which is of dimension two. In fact, the analysis of its wave function shows that $\langle 5_1^+ | B^3[12]5 \rangle^2 = 0.961$ (0.967) in ^{46}V (^{50}Mn). The 5_1^+ state can therefore be written approximately as $|B^3[12]5\rangle$, which is nothing but the normalized B -pair state (1) with $n = 3$, $L_2 = 12$ and $L_3 = 5$, and the structure of this state is now understood in simple terms. For example, within this approximation its energy can be given as

$$E(B^3[12]5) = \frac{7695}{11668} v_{01}^{2f} + \frac{564181}{665076} v_{10}^{2f} + \frac{6112703}{1551844} v_{21}^{2f} + \frac{2544169}{2438612} v_{30}^{2f} + \frac{5814660}{4267571} v_{41}^{2f} + \frac{3323}{5834} v_{50}^{2f} + \frac{3705457}{1219306} v_{61}^{2f} + \frac{340651}{96261} v_{70}^{2f}, \quad (43)$$

in terms of the shell-model two-body matrix elements v_{JT}^{2f} . One notes the large coefficient in front of the ‘quadrupole pairing’ matrix element v_{21}^{2f} . Quadrupole collectivity will therefore strongly influence the energy of the 5_1^+ level, in both ^{46}V and ^{50}Mn .

The derivation of the energy formula (43) is non-trivial since it requires a symbolic implementation of Chen’s recursive algorithm,³² and overlaps involving up to four pairs are needed [see Eq. (15)]. On the basis of more ‘elementary’ techniques, an approximate formula can be obtained as follows. For a three-boson state, its diagonal energy originating from a two-body interaction can be written with CFPs,

$$V(b^3[L_2]L_3) = 3 \sum_{L_2'} [\ell^2(L_2)\ell] \ell^3[L_2]L_3]^2 v_{L_2'}^{2b}, \quad (44)$$

which are known in closed form in terms of Racah coefficients,⁴⁰ leading to the expression

$$V(b^3[12]5) = \frac{4370}{11557} v_2^{2b} + \frac{512325}{392938} v_4^{2b} + \frac{12650}{41021} v_6^{2b} + \frac{300}{31369} v_8^{2b} + \frac{405}{12265279} v_{10}^{2b} + \frac{14859}{14858} v_{12}^{2b}. \quad (45)$$

Since the boson interaction matrix elements v_L^{2b} are known in terms of the two-body fermion matrix elements v_{JT}^{2f} from Eq. (40), the following total energy (which

includes the single-boson energy $3\epsilon_b$) is found:

$$\begin{aligned} E(b^3[12]5) &= 3\epsilon_b + V(b^3[12]5) \\ &= 0.891 v_{01}^{2f} + 0.888 v_{10}^{2f} + 3.637 v_{21}^{2f} + 0.981 v_{30}^{2f} + \\ &\quad 1.416 v_{41}^{2f} + 0.592 v_{50}^{2f} + 3.056 v_{61}^{2f} + 3.540 v_{70}^{2f}, \end{aligned} \quad (46)$$

where the coefficients are rational numbers involving very large integers, to which a numerical approximation is given.

Equation (46) is the boson analogue of the shell-model result (43). The expressions are similar but not identical, and this is due to the two-body approximation in the boson calculation. It should be emphasized once more that, if three-body interactions between the bosons are included, results in the B^3 and b^3 spaces become identical.

The nuclei ^{46}V and ^{50}Mn have several isomeric states,⁴² with half-lives ranging from minutes (the 5_1^+ level in ^{50}Mn) to milli- and nano-seconds (the 3_1^+ and 5_1^+ , 7_1^+ levels in ^{46}V , respectively), some of which have known dipole and/or quadrupole moments. The measured magnetic dipole moments can be compared with the simple single- j shell prediction that the g factor of any state in an $N = Z$ nucleus equals $(g_\nu + g_\pi)/2$ (see Subsect. 7.1). The effect of the quenching of the spin part of the M1 operator is small: without quenching $(g_\nu + g_\pi)/2$ equals $0.55 \mu_N$ in the $1f_{7/2}$ shell and, with a quenching of 0.7, it reduces to $0.52 \mu_N$. Therefore, the single- j shell model predicts magnetic dipole moments $\mu(J^\pi)$ of states in $N = Z$ nuclei in the $1f_{7/2}$ shell of the order of $0.52J$ to $0.55J \mu_N$. This agrees with the measured^{45,46} values of $\mu(3_1^+) = 1.64$ (3) μ_N in ^{46}V and $\mu(5_1^+) = 2.76$ (1) μ_N in ^{50}Mn . As argued in Subsect. 7.1, this result does not constitute a test of the B -pair or b -boson approximation, but shows consistency with a single- j shell truncation. The large-scale shell-model result with the gxpfla interaction,⁴⁶ $\mu(5_1^+) = 2.81 \mu_N$, also agrees with the data.

Charlwood *et al.*⁴⁶ also measured the quadrupole moment of the 5^+ isomer in ^{50}Mn , $Q(5_1^+) = +0.80$ (12) b. In the large-scale shell model with the gxpfla interaction⁴⁶ one finds a smaller value of $Q(5_1^+) = +0.58$ b. A numerical calculation in a single- j shell gives $Q(5_1^+) = +4.2(e_\nu + e_\pi)l_{\text{ho}}^2$, in terms of the neutron (proton) effective charges e_ν (e_π) and the oscillator length l_{ho} of Eq. (34). This shell-model result can be compared with the approximation in terms of b bosons that assumes $|5_1^+\rangle \approx |b^3[12]5\rangle$. One uses the definition

$$Q(b^3[L_2]L_3) = \sqrt{\frac{16\pi}{5}} \begin{pmatrix} L_3 & 2 & L_3 \\ -L_3 & 0 & L_3 \end{pmatrix} \langle b^3[L_2]L_3 || e_b(b^\dagger \times \tilde{b})^{(2)} || b^3[L_2]L_3 \rangle, \quad (47)$$

where the reduced matrix element is obtained from

$$\begin{aligned} \langle b^3[L_2]L_3 || (b^\dagger \times \tilde{b})^{(\lambda)} || b^3[L'_2]L'_3 \rangle &= 3(-)^{\ell+L_3+\lambda} \sqrt{(2\lambda+1)(2L_3+1)(2L'_3+1)} \\ &\quad \times \sum_{L''_2} [\ell^2(L''_2)\ell] \ell^3[L_2]L_3 [\ell^2(L''_2)\ell] \ell^3[L'_2]L'_3 \begin{Bmatrix} \ell & L_3 & L''_2 \\ L'_3 & \ell & \lambda \end{Bmatrix}. \end{aligned} \quad (48)$$

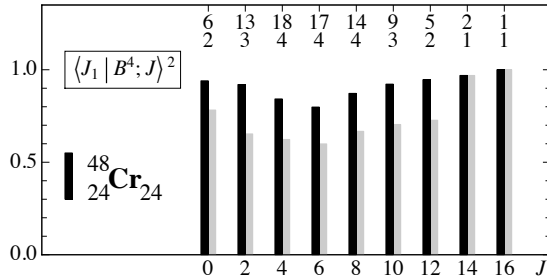


Fig. 9. The square of the projection of the yrast eigenstates in the $(1f_{7/2})^8$ system onto the subspace spanned by the B -pair states $|B^4; J\rangle$, for angular momentum J and isospin $T = 0$. The shell-model interaction is defined in Eq. (39). Also shown are the numbers of $(1f_{7/2})^8$ states (top) and of B -pair states (bottom) with angular momentum J and isospin $T = 0$. The grey bars represent the corresponding analysis in terms of the stretched configuration of Danos and Gillet^{28,29} (see text).

For $L_2 = 12$ and $L_3 = 5$ and with the effective boson charge taken from Eq. (38), one finds

$$Q(b^3[12]5) = -\frac{649485}{150241}(e_\nu + e_\pi)l_{\text{ho}}^2 \approx -4.3(e_\nu + e_\pi)l_{\text{ho}}^2, \quad (49)$$

in excellent agreement with the single- j shell result, considering that a change of sign of the quadrupole moment is needed to pass from particles to holes.⁴⁴

In a single- j shell calculation the quadrupole deformation is significantly underestimated if standard values for the effective charges ($e_\nu = 0.5$ and $e_\pi = 1.5$) are taken. The dependence of the quadrupole moments on effective charges and on the oscillator length can be eliminated by considering ratios. For example, by making the associations $|7_1^+\rangle = |b\rangle$ and $|5_1^+\rangle \approx |b^3[12]5\rangle$ for the 7_1^+ and 5_1^+ states in ${}^{42}\text{Sc}$ and ${}^{46}\text{V}$, respectively, one obtains the ratio

$$\frac{Q(5_1^+; {}^{46}\text{V})}{Q(7_1^+; {}^{42}\text{Sc})} \approx \frac{Q(b^3[12]5)}{Q(b)} = \frac{216495}{150241} \approx 1.44. \quad (50)$$

This is a parameter-independent test of the validity of the b -boson approximation.

7.2.3. ${}^{48}\text{Cr}$

The B -pair content of yrast states in ${}^{48}\text{Cr}$ is displayed in Fig. 9 while its energy spectrum, calculated in various approximations, single- j shell model (SM), shell-model B -pair approximation (B^4) and mapped b -boson calculation with up to two-body ($b^4[2]$) and three-body ($b^4[3]$) interactions, is shown in Fig. 10.

Two issues of interest arise for the eight-nucleon system. First, it is possible to establish an explicit connection with the stretch scheme of Danos and Gillet,^{28,29} since their eight-nucleon stretched state $|B_s^4 J\rangle$ with angular momentum J can in

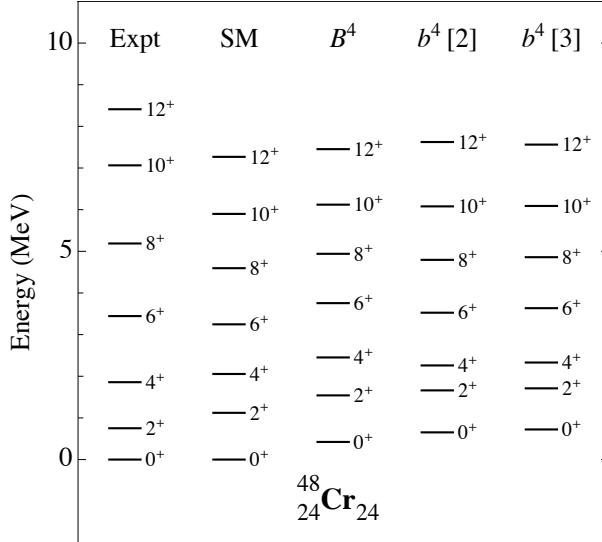


Fig. 10. The yrast spectrum of ${}^{48}\text{Cr}$. Levels are labelled by their angular momentum and parity J^π . The different columns contain the experimental⁴² levels (Expt), the results of the $(1f_{7/2})^8$ shell model (SM) with the interaction (39), the lowest eigenvalue of the shell-model hamiltonian in the B -pair subspace (B^4) and the lowest eigenvalue of the mapped boson hamiltonian with up to two-body ($b^3[2]$) and up to three-body ($b^3[3]$) interactions. The shell-model energy of the 0_1^+ level is normalized to zero.

fact be written as

$$\begin{aligned}
 |B_s^4 J\rangle &\propto \left((B^\dagger \times B^\dagger)^{(J_{\max})} \times (B^\dagger \times B^\dagger)^{(J_{\max})} \right)^{(J)} |0\rangle \\
 &= (-)^J \sum_L \sqrt{(2J_{\max} + 1)(2L + 1)} \left\{ \begin{matrix} J_{\max} & \ell & L \\ \ell & J & J_{\max} \end{matrix} \right\} |B^4 J_{\max} L J\rangle, \quad (51)
 \end{aligned}$$

in terms of the B -pair states (1) with $J_{\max} = 4j - 2$. It is therefore possible to determine the ‘stretch’ content of a shell-model state since it can be done for the states on the right-hand side of Eq. (51) with the formalism developed in Sect. 4. Note that, unlike in the original discussion of Danos and Gillet,^{28,29} anti-symmetry of the stretched configuration is fully taken into account here. The stretch content of yrast states in ${}^{48}\text{Cr}$ is shown with grey bars in Fig. 9. It is clear from Eq. (51) that the stretched configuration is but one particular vector in the B -pair space and the stretch content of any state is therefore necessary smaller than its B -pair content. If the dimension of the B -pair space reduces to one, as is the case for $J = 14$ and 16 , both approximations become identical. These findings are completely at variance with the results of Daley.³⁰

The second question of interest concerns the formation of a B -pair condensate. The ground state lies dominantly in the B -pair space and can, to a good approximation, be written in terms of a single component $|B^2[0]B^2[0]; 0\rangle$ which

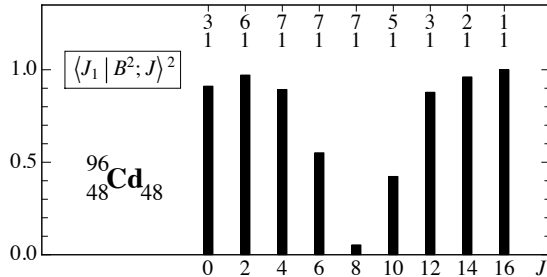


Fig. 11. Overlaps of the yrast eigenstates in the $(1g_{9/2})^4$ system, for angular momentum J and isospin $T = 0$, with the B -pair state $|B^2; J\rangle$. The shell-model interaction SLGT0 is defined in Table 1. Also shown are the numbers of $(1g_{9/2})^4$ states (top) and of B -pair states (bottom) with angular momentum J and isospin $T = 0$.

arises by the coupling of *pairs of B pairs* to angular momentum zero. A wave-function analysis shows that $\langle 0_1^+ | B^2[0]B^2[0]; 0 \rangle^2 = 0.927$, close to full B -pair content of 0.940. A similar approximation is possible for the 2_1^+ state for which $\langle 2_1^+ | B^2[0]B^2[2]; 2 \rangle^2 = 0.918$. In view of these large overlaps, it is then tempting to postulate a seniority-like scheme for the B pairs [and therefore an $SO(2\ell + 1)$ classification for the b bosons] but this would be wrong. Although the 4_1^+ state has a dominant B -pair content (84.2%), its B -pair seniority-like component is negligible, $\langle 4_1^+ | B^2[0]B^2[4]; 4 \rangle^2 = 0.005$. The two-body boson interactions, derived from the shell model and shown in Table 3, do not allow an obvious treatment in terms of boson symmetries. It remains nevertheless true that the single component $|B^2[0]B^2[0]; 0\rangle$ provides a good approximation to the *ground state* of the eight-nucleon system. It would be of some interest to generalize this finding to larger single- j shells and to many particles.

7.3. The $1g_{9/2}$ shell

Nuclei in the $1g_{9/2}$ shell were the focus of a previous study,⁴⁸ with a wave-function analysis limited to $n = 4$ and boson interactions limited to two-body. Additional material and further details of the calculations are presented in the subsequent subsections. The shell-model interaction, referred to as SLGT0, is taken from Serduke *et al.*⁴⁹ and gives satisfactory results for the neutron-deficient nuclei in the mass region $A = 86$ to 100.⁵⁰ Being defined in the $2p_{1/2} + 1g_{9/2}$ shell-model space, this interaction is renormalized to the $1g_{9/2}$ orbit. The resulting matrix elements are given in Table 1.

7.3.1. ^{96}Cd

The study of this nucleus is at the limit of present experimental capabilities. A fusion–evaporation experiment was proposed at GANIL some time ago⁵¹ but had

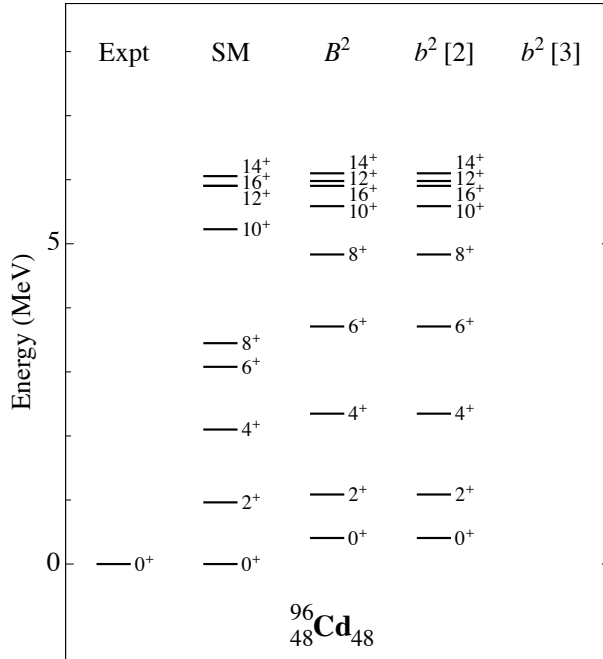


Fig. 12. The yrast spectrum of ^{96}Cd . Levels are labelled by their angular momentum and parity J^π . No experimental levels are known to date. The different columns contain the results of the $(1g_{9/2})^4$ shell model (SM) with the SLGT0 interaction, the expectation value of the shell-model hamiltonian in the B -pair state (B^2) and the expectation value of the mapped boson hamiltonian with up to two-body interactions ($b^2[2]$). The shell-model energy of the 0_1^+ level is normalized to zero.

to be rescheduled to due to technical difficulties. In view of this current interest, it is worthwhile to investigate the B -pair structure of ^{96}Cd . The B -pair content of shell-model states calculated with the SLGT0 interaction is shown in Fig. 11. Results are entirely consistent with those obtained in the $1f_{7/2}$ shell (see Fig. 3), indicating the generic nature of the analysis, independent of the particular value of j of the shell considered. The decrease of the B -pair content at intermediate values of the angular momentum J can be understood on the basis of a combination of geometry—the CFPs in a single- j shell, and dynamics—the dependence of the interaction matrix elements on J and T .⁴⁸

The energy spectrum of ^{96}Cd , calculated in various approximations, single- j shell model (SM), shell-model B -pair approximation (B^2) and mapped b -boson calculation ($b^2[2]$), is shown in Fig. 12. Results are seen to be consistent with the wave-function analysis. The boson interaction matrix elements v_L^{2b} are known analytically in terms of the two-body fermion matrix elements v_{JT}^{2f} [see Eq. (40)] with universal coefficients $a_{JT}^I(j = 9/2)$ given in Table 4. The resulting boson interaction matrix elements are shown in Table 3. There is no four-particle shell-model state

Table 4. Coefficients $a_{JT}^L(j)$ in the expansion (40) for $j = 9/2$.

(JT)	L	0	2	4	6	8	10	12	14	16
(01)		4862 4265	117572 137755	2261 5445	7429 65120	1311 110210				
(10)		35802 46915	976752 1515305	2261 5445	11799 65120	50301 1212310	345 156739			
(21)		15912 9383	543932 303061	13889 7986	45011 35816	67068 121231	15525 156739			
(30)		139944 609895	6520724 19698965	771799 1557270	165669 291005	3425436 7880015	359415 2037607	1218 69355		
(41)		99144 609895	6750054 19698965	618032 778635	821583 582010	13883076 7880015	2774250 2037607	63423 138710		
(50)		408 55445	43358 1790815	686 7865	483 2035	334476 716365	578322 926185	29957 63050	868 8515	
(61)		153 93830	13699 1515305	343 6655	76797 358160	789021 1212310	1113966 783695	109881 53350	1953 1310	
(70)		81 4147286	13608 66976481	1673 882453	14823 1217744	3024621 53584102	6727847 34639319	1148337 2358070	46251 57902	8 17
(81)		1 1219790	378 19698965	161 519090	29223 9312160	175176 7880015	1202172 10188035	15231 31525	1977 1310	3
(90)		1 24310	8081 334882405	23 802230	101 14391520	61018 133960255	479646 173196595	10893 535925	2211 22270	9 17

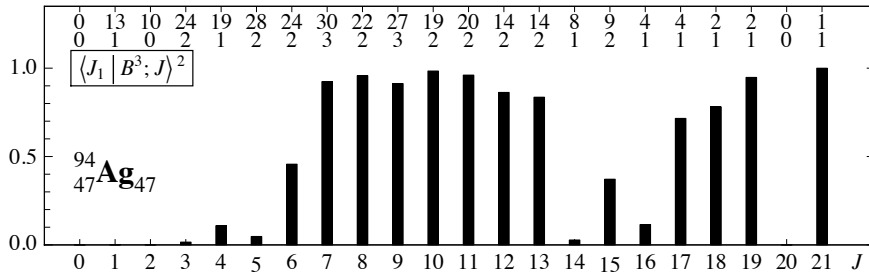


Fig. 13. The square of the projection of the yrast eigenstates in the $(1g_{9/2})^6$ system onto the subspace spanned by the B -pair states $|B^3; J\rangle$, for angular momentum J and isospin $T = 0$. The shell-model interaction SLGT0 is defined in Table 1. Also shown are the numbers of $(1g_{9/2})^6$ states (top) and of B -pair states (bottom) with angular momentum J and isospin $T = 0$.

with $J = 18$, implying the choice $v_{18}^{2b} = +\infty$.

7.3.2. ^{94}Ag

Not much is known experimentally about ^{94}Ag except for the presence of two isomers, with tentative assignments $J^\pi = 7^+$ (presumably the lowest $T = 0$ state) and $J^\pi = 21^+$, the latter at 6.7 (5) MeV above the 0^+ ground state.^{52,53} The subsequent discussion is focussed on the structure of these two states.

The B -pair content is obtained from Eq. (12) with ω up to 3, the maximum dimension of the B -pair space (for $J = 7$ and 9). This quantity is shown in Fig. 13 for yrast states in ^{94}Ag , together with the dimensions of the shell-model and B -pair spaces. The results are in total accord with those found in the $1f_{7/2}$ shell (see Fig. 6), considering that the replacement $j = 7/2 \rightarrow 9/2$ leads to an overall increase

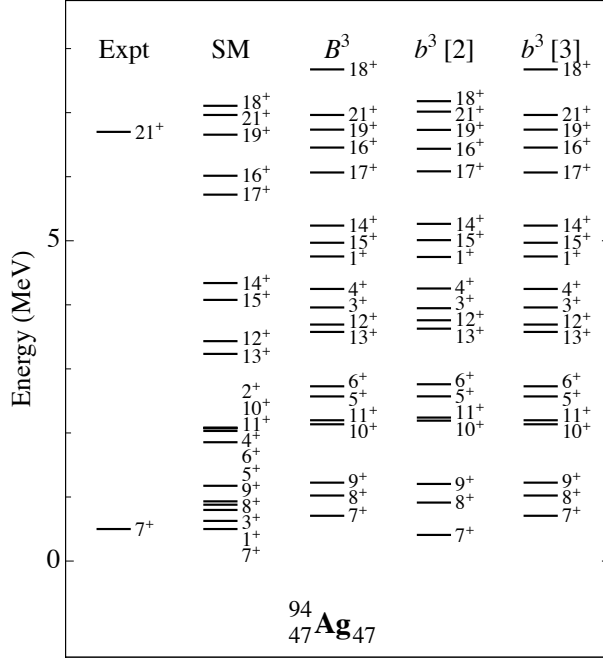


Fig. 14. The spectrum of yrast states in ^{94}Ag with isospin $T = 0$. Levels are labelled by their angular momentum and parity J^π . The different columns contain the experimental^{52, 53} levels (Expt), the results of the $(1g_{9/2})^6$ shell model (SM) with the SLGT0 interaction, the lowest eigenvalue of the shell-model hamiltonian in the B -pair subspace (B^3) and the lowest eigenvalue of the mapped boson hamiltonian with up to two-body ($b^3[2]$) and up to three-body ($b^3[3]$) interactions. The shell-model energy of the $T = 0$ ground state, $J^\pi = 7^+$, is normalized to the experimental excitation energy of this level which is relative to the 0^+ ground state.

of the angular momenta involved. It is seen in particular that the overlaps are high for $J = 21$ (which is trivial) and for $J = 7$ (which is not), making the analysis of these states in terms of B pairs or b bosons meaningful.

The energy spectrum of $T = 0$ states in ^{94}Ag is shown in Fig. 14. The shell model with the SLGT0 interaction gives the correct $T = 0$ ground-state spin, $J = 7$, and the energy of the 21^+ isomer comes out reasonably close to its observed value. The B -pair calculation agrees with the shell model but for the low-spin states (1^+ to 5^+) which are obtained at much higher energies.

In a shell-model description where three neutrons and three protons are placed in the $1g_{9/2}$ orbit, the $J = 21$ state is stretched and therefore unique. In this single- j shell approximation, the 21^+ isomer can therefore be written exactly as $|B^3[12]21\rangle$, the normalized B -pair state (1) with $n = 3$, $L_2 = 12$ and $L_3 = 21$. Chen's algorithm³² then provides the following energy expression for this state:

$$E(B^3[12]21) = \frac{21}{65}v_{50}^{2f} + \frac{21}{10}v_{61}^{2f} + \frac{645}{442}v_{70}^{2f} + \frac{69}{10}v_{81}^{2f} + \frac{717}{170}v_{90}^{2f}, \quad (52)$$

in terms of the two-body fermion matrix elements v_{JT}^{2f} . This energy expression can alternatively (and more simply) be derived with standard techniques involving CFPs. Since a six-nucleon state with angular momentum $J_6 = 21$ and isospin $T_6 = 0$ is unique, its energy is given as

$$E(j^6 J_6 = 21, T_6 = 0) = \sum_{JT} a_{JT} v_{JT}^{2f}, \quad (53)$$

with the coefficients a_{JT} known in terms of $6 \rightarrow 4$ CFPs,⁴⁰

$$a_{JT} = 15 \sum_{\alpha_4 J_4 T_4} [j^4(\alpha_4 J_4 T_4) j^2(JT)] \{j^6 J_6 = 21, T_6 = 0\}^2. \quad (54)$$

It can be verified that this alternative derivation also yields the expression (52), which provides a rigorous check on the correctness of the implementation of Chen's algorithm. It should be emphasized that the derivation using standard CFP techniques is valid only for shell-model states that are unique, such as the 21^+ isomer. If several states can be constructed for a given J and T , no such derivation is possible while an expression still can be found from B pairs, as illustrated below for the 7^+ isomer.

In terms of bosons, the 21^+ isomer arises from the coupling of three b bosons with spin $\ell = 9$ to total angular momentum $J = 21$. It can be shown⁵⁴ that *two* independent boson states exist with $J = 21$, one of which must be spurious. Let us consider this case in detail, to illustrate the mechanism by which spurious states can be eliminated analytically. The two independent boson states may be chosen as $|b^3[\tilde{1}2]21\rangle$ and $|b^3[\tilde{1}4]21\rangle$, assumed to be normalized and orthogonal. Since the CFPs needed in a three-particle problem are known in terms of Racah coefficients,⁴⁰ the energy matrix can be shown to have the following elements:

$$\begin{aligned} \langle b^3[\tilde{1}2]21 | \hat{H}_2^b | b^3[\tilde{1}2]21 \rangle &= 3\epsilon_b + \frac{2833}{2697} v_{12}^{2b} + \frac{44200}{263469} v_{14}^{2b} + \frac{2782494}{2546867} v_{16}^{2b} + \frac{60536}{87823} v_{18}^{2b}, \\ \langle b^3[\tilde{1}4]21 | \hat{H}_2^b | b^3[\tilde{1}4]21 \rangle &= 3\epsilon_b + \frac{3337047}{1932106} v_{14}^{2b} + \frac{82824}{87823} v_{16}^{2b} + \frac{20553}{62326} v_{18}^{2b}, \\ \langle b^3[\tilde{1}2]21 | \hat{H}_2^b | b^3[\tilde{1}4]21 \rangle &= \sqrt{\frac{24582912900}{84841672619}} v_{14}^{2b} - \sqrt{\frac{7946802864}{7712879329}} v_{16}^{2b} + \sqrt{\frac{20067684}{88284779}} v_{18}^{2b}, \end{aligned} \quad (55)$$

where ϵ_b is the energy of the b boson. The stretched boson interaction matrix element v_{18}^{2b} appears in the diagonal and the off-diagonal matrix elements and, consequently, in the limit $v_{18}^{2b} \rightarrow +\infty$, one eigenvalue of the 2×2 matrix (55) tends to infinity while the lowest eigenvalue acquires the expression

$$E(b_\infty^3 21) = 3\epsilon_b + \frac{6851}{20155} v_{12}^{2b} + \frac{15488}{21545} v_{14}^{2b} + \frac{1212882}{624805} v_{16}^{2b}, \quad (56)$$

where the index ' ∞ ' serves as a reminder of the limit procedure used to derive the result. The procedure also yields the components of the state,

$$|b_\infty^3 21\rangle = -\sqrt{\frac{637143}{1968935}} |b^3[\tilde{1}2]21\rangle + \sqrt{\frac{1331792}{1968935}} |b^3[\tilde{1}4]21\rangle, \quad (57)$$

of use in the calculation of the quadrupole moment of the 21^+ isomer (see below). Since the b -boson energy and the boson interaction are known in terms of the two-body fermion interaction, Eq. (56) can be converted into

$$E(b_\infty^3 21) = \frac{22134}{3707825} v_{30}^{2f} + \frac{1152549}{7415650} v_{41}^{2f} + \frac{1347751953}{5740387250} v_{50}^{2f} + \frac{8606149749}{4857250750} v_{61}^{2f} + \frac{354940047213}{214690483150} v_{70}^{2f} + \frac{1561553973}{220784125} v_{81}^{2f} + \frac{15411107094}{3753330125} v_{90}^{2f}. \quad (58)$$

This is an approximate expression since it is derived by use of a mapping that includes up to two-body interactions between the bosons. To what extent it is wrong therefore yields an idea about the reliability of the two-body boson mapping. Since the highest allowed angular momentum for two neutrons and two protons in a $j = 9/2$ orbit is $J = 16$, only matrix elements v_{JT}^{2f} with $J \geq 5$ can contribute to the energy of the $J = 21$ state. This rule is obviously obeyed in Eq. (52) but violated in Eq. (58). It is seen, however, that the coefficients of v_{30}^{2f} and v_{41}^{2f} are rather small in the latter expression, indicating that the two-body boson approximation is reasonably accurate.

The perplexed reader might well wonder what could be the purpose of quoting in Eq. (58) the coefficients a_{JT} as the ratio of two ridiculously large integers. The advantage of the use of *exact* numbers is that enables a rigorous check of fermionic as well as bosonic calculations. The coefficients a_{JT} in an energy expression $E(j^n J_n T_n) = \sum_{JT} a_{JT} v_{JT}^{2f}$ for a unique n -particle shell-model state j^n with total angular momentum J_n and isospin T_n , satisfy the identities

$$\begin{aligned} \sum_{JT} a_{JT} &= \frac{n(n-1)}{2}, \\ \sum_{JT} J(J+1) a_{JT} &= J_n(J_n+1) + j(j+1) \times n(n-2), \\ \sum_{JT} T(T+1) a_{JT} &= T_n(T_n+1) + \frac{3}{4} n(n-2). \end{aligned} \quad (59)$$

These identities reflect the conservation of particle number, angular momentum and isospin in the shell model and are therefore valid for the coefficients in Eq. (52). They are also *exactly* satisfied by the coefficients in Eq. (58). This is a consequence of the preservation of n , J and T under the mapping procedure.

The yrast 7^+ state in ^{94}Ag is the analogue of the 5^+ state in ^{46}V or ^{50}Mn , discussed in Subsect. 7.2.2. Its structure is particularly simple since a wave-function analysis shows that $\langle 7_1^+ | B^3[16]7 \rangle^2 = 0.908$. The 7^+ isomer is now understood in simple terms as it results from the coupling of two B pairs to maximal angular momentum $J = 16$ ($J = 18$ is not allowed by the Pauli principle) which is subsequently coupled with the third B pair to total $J = 7$. Within this approximation its energy is calculated as

$$E(B^3[16]7) = 0.528 v_{01}^{2f} + 0.654 v_{10}^{2f} + 3.302 v_{21}^{2f} + 1.081 v_{30}^{2f} + 1.977 v_{41}^{2f} + 0.256 v_{50}^{2f} + 0.190 v_{61}^{2f} + 0.480 v_{70}^{2f} + 3.002 v_{81}^{2f} + 3.529 v_{90}^{2f}. \quad (60)$$

For the SLGT0 interaction this formula gives an energy of -11.069 MeV, to be compared with a correlation energy of -11.276 MeV if the full $(1g_{9/2})^6$ shell-model basis is used. Note also that the approximate energy expression for the 7_1^+ state in the $(1g_{9/2})^6$ system is similar to the one obtained in Eq. (43) for the 5_1^+ state in the $(1f_{7/2})^6$ system.

Since the dimension of the B -pair space is three and equals the number of independent states for three b bosons with spin $\ell = 9$, no spurious boson states occur for $J = 7$. The calculation of the energy of the three-boson state $|b^3[16]7\rangle$ in a two-body boson approximation is then straightforward and proceeds along the lines of the energy calculation for the 5^+ state in Subsect. 7.2.2 [see Eqs. (44), (45) and (46)], leading to the expression

$$E(b^3[16]7) = 0.711v_{01}^{2f} + 0.711v_{10}^{2f} + 3.118v_{21}^{2f} + 0.997v_{30}^{2f} + 1.933v_{41}^{2f} + 0.278v_{50}^{2f} + 0.235v_{61}^{2f} + 0.484v_{70}^{2f} + 3.004v_{81}^{2f} + 3.529v_{90}^{2f}, \quad (61)$$

in close correspondence with the fermion result (60) that takes into account the exchange terms between the B pairs.

The discussion of Subsect. 7.1 concerning magnetic dipole moments also applies to the $1g_{9/2}$ shell. The single- j shell prediction for the g factor of any state in a $1g_{9/2}$ $N = Z$ nucleus is $0.54 \mu_N$ without spin quenching and $0.51 \mu_N$ with a quenching of 0.7. Magnetic dipole moments do not provide a test of the B -pair or b -boson approximation but measured μ values that deviate from the narrow range predicted in a single- j shell, would be indicative of admixtures of configurations beyond the $1g_{9/2}$ shell.

It is also of interest to predict the quadrupole moments of the isomeric states in ^{94}Ag . The shell-model calculation in a single- j approximation can be worked analytically for $J = 21$ since the state is unique,

$$Q(B^3[12]21) = -\sqrt{\frac{196}{6}}(e_\nu + e_\pi)l_{\text{ho}}^2 \approx -0.42 \text{ b}. \quad (62)$$

The corresponding boson result is obtained from the expansion (57), together with the general expressions (47) and (48), leading to

$$Q(b_\infty^3 21) = -\sqrt{\frac{81949367824}{3489855625}}(e_\nu + e_\pi)l_{\text{ho}}^2 \approx -0.44 \text{ b}, \quad (63)$$

which illustrates the reliability of the boson approximation. The shell-model value of the quadrupole moment of the 7^+ isomer can be obtained numerically and, in a single- j shell approximation, gives $Q(7_1^+) = +6.60(e_\nu + e_\pi)l_{\text{ho}}^2 \approx +0.60 \text{ b}$. The dominant component of this state in terms of b bosons is $|b^3[16]7\rangle$ for which, from Eqs. (47) and (48), the following quadrupole moment is found:

$$Q(b^3[16]7) = -\sqrt{\frac{30930277300923364}{627253477610841}}(e_\nu + e_\pi)l_{\text{ho}}^2 \approx -0.64 \text{ b}. \quad (64)$$

The quadrupole moments (62), (63) and (64) are given for particle-particle configurations; an additional sign is needed to pass to the hole-hole nucleus ^{94}Ag .

A final word is needed on the nature of boson approximation. Consider as an example the matrix element

$$\langle B^3[4]7|B^3[16]7\rangle = \sqrt{\frac{112919600563049280}{139849953265085321}} \approx 0.899, \quad (65)$$

where it is assumed that bra and ket states are normalized but evidently non-orthogonal. As can be expected from a fraction which involves very large integers, the calculation of this overlap is non-trivial. The corresponding boson result,

$$\langle b^3[4]7|b^3[16]7\rangle = \sqrt{\frac{7012200}{8733503}} \approx 0.896, \quad (66)$$

is obtained much more simply in terms of $3 \rightarrow 2$ CFPs associated with bosons with spin $\ell = 9$. The reliability of the mapping of B pairs onto b bosons ultimately is due to the negligible effect of exchange terms between the B pairs. It cannot be emphasized enough that the calculation of matrix elements of the type (65) is highly non-trivial and quickly runs into computational problems as the number of pairs increases. By comparison, the calculation of overlaps of the type (66) is trivial and can easily be done for all cases of relevance.

7.3.3. ^{92}Pd

The low-lying yrast states of this nucleus were measured by Cederwall *et al.*²⁵ with the aim to probe the importance of aligned neutron–proton pairs in $N \sim Z$ nuclei. An analysis of shell-model wave functions in terms of B pairs, as performed for all $N = Z$ nuclei previously considered, becomes tedious in this case, owing to the dimensions of complete bases $|\bar{P}_r^n\rangle$ in terms of pairs $P_{\Gamma M_T}^\dagger$. The complete study of the $1f_{7/2}$ shell presented in Subsect. 7.2 and the results obtained so far for the $1g_{9/2}$ shell indicate that an analysis of ^{92}Pd in the B -pair subspace and a subsequent mapping to b bosons should give meaningful results.

The energy spectrum of ^{92}Pd , calculated in various approximations, single- j shell model (SM), shell-model B -pair approximation (B^4) and mapped b -boson calculation with up to two-body ($b^4[2]$) and three-body ($b^4[3]$) interactions, is shown in Fig. 15. The B -pair calculation shows an underbinding of about 0.8 MeV. This feature is considerably improved if S pairs or s bosons are added to the basis.⁴⁸

A final illustration of the b -boson approximation is provided in Fig. 16 where E2 transition strengths between yrast ($1g_{9/2}$)⁸ states as calculated in the shell model are compared with those obtained with b bosons. The shell-model reduced matrix elements are calculated with the operator (34) and expressed in units $(e_\nu + e_\pi)l_{\text{ho}}^2$. The reduced matrix elements in the boson approximation are calculated with the operator (36) with an effective boson charge e_b , determined from the neutron and proton effective charges according to Eq. (37). A small depletion of the shell-model E2 strength is perceptible for $J \approx 8$ and is absent in the boson calculation. Apart from this deviation both calculations agree, indicating once more that the shell-model wave functions can be adequately represented in terms of a single b boson.

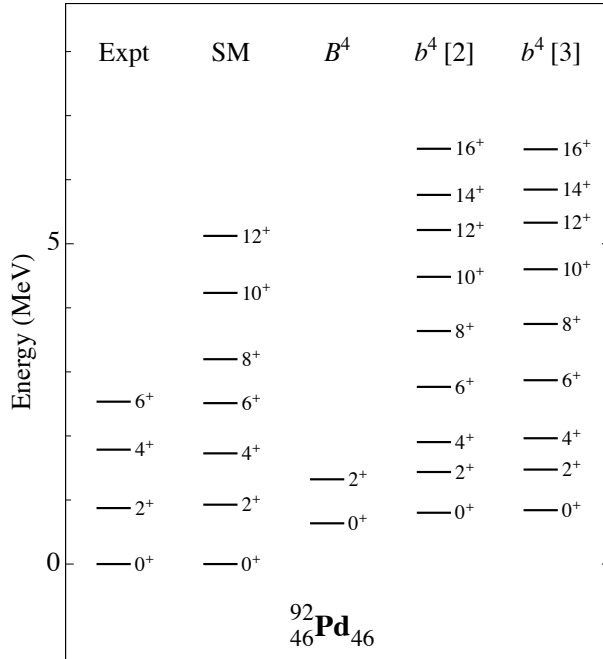


Fig. 15. The yrast spectrum of ${}^{92}\text{Pd}$. Levels are labelled by their angular momentum and parity J^π . The different columns contain the experimental²⁵ levels (Expt), the results of the $(1g_{9/2})^8$ shell model (SM) with the SLGT0 interaction, the lowest eigenvalue of the shell-model hamiltonian in the B -pair subspace (B^4) and the lowest eigenvalue of the mapped boson hamiltonian with up to two-body ($b^4[2]$) and up to three-body ($b^4[3]$) interactions. The shell-model energy of the 0_1^+ level is normalized to zero.

It should be emphasized that, although the number of B -pair states is but a small subset of the total number of possible shell-model states, no effective boson charge is needed to arrive at the agreement found in Fig. 16.

8. Conclusions

What can be concluded with regard to the three approximations enounced in Sect. 6? (i) Can the shell-model space be truncated to a single high- j orbit? (ii) Can the single- j shell space be reduced to one written in terms of aligned B pairs? (iii) And, finally, can the aligned B pair be replaced by a b boson? The answer to the question (iii) is unreservedly positive: owing to its high angular momentum, the B pair behaves much as a boson. The mapping from B -pair to b -boson space can be made exact by including appropriate interactions between the bosons but becomes approximate if higher-order interactions are neglected. The examples of the $1f_{7/2}$ and $1g_{9/2}$ shells show that two-body interactions between the bosons suffice and that no higher-order interactions are needed. The answer to question (ii) is positive for most but not for all yrast states, that is, most but not all yrast states of $N = Z$

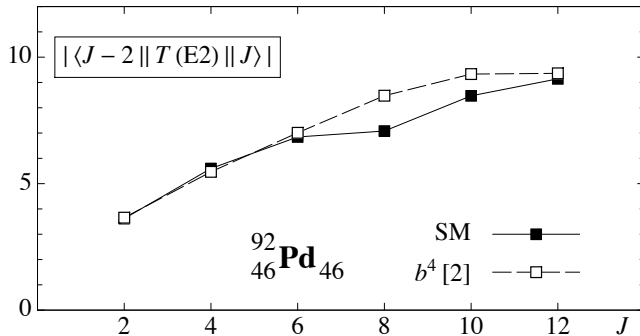


Fig. 16. Absolute values of the reduced matrix elements of the E2 operator for the transitions $J \rightarrow J - 2$ between yrast states with $T = 0$ (^{92}Pd), calculated in the $(1g_{9/2})^8$ shell model (SM) with the SLGT0 interaction and compared with the mapped b -boson calculation ($b^4[2]$). Matrix elements are expressed in units $(e_\nu + e_\pi)l_{\text{ho}}^2$.

nuclei can be written in terms of B pairs. Odd-odd $N = Z$ nuclei in particular behave in a schizophrenic manner with only a subset of their yrast states having a sizeable B -pair content. Other states, mostly of low angular momentum, require the inclusion of low-spin pairs such as those mapped onto the corresponding bosons of the IBM-4. In almost all cases, however, a simple interpretation can be given of yrast states in terms of neutron-proton pairs and this enables one to intuit the complex spectroscopy of odd-odd $N = Z$ nuclei and to derive simple parameter-free predictions. The validity of the truncation to a single- j shell depends on specific features of a realistic shell-model hamiltonian and the answer to question (i) may therefore be different for the $1f_{7/2}$ and $1g_{9/2}$ shells considered in this review. A recent large-scale shell-model calculation with a realistic effective interactions seems to indicate that the truncation to $1g_{9/2}$ (and therefore the B -pair approximation) is justified in the $A \sim 90$ – 100 region.³⁵ But, in the end, only the experimental verification of the simple predictions derived on the basis of the b -boson approximation will be able to tell whether $N = Z$ nuclei exist with sufficiently isolated single- j shells.

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