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A unified approach to SIC-POVMs and MUBs

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Abstract

A unified approach to (symmetric informationally complete) positive operator valued measures and mutually unbiased bases is developed in this article. The approach is based on the use of operator equivalents expanded in the enveloping algebra of SU(2). Emphasis is put on similarities and differences between SIC-POVMs and MUBs.

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1 INTRODUCTION

The importance of finite-dimensional spaces for quantum mechanics is well recognized (see for instance Refs. [1, 2]). In particular, such spaces play a major role in quantum information theory, especially for quantum cryptography and quantum state tomography [3-21]. Along this vein, a symmetric informationally complete (SIC) positive operator valued measure (POVM) is a set of operators acting on a finite Hilbert space [3-11] and mutually unbiased bases (MUBs) are specific bases for such a space [12-21].

The introduction of a POVM goes back to the seventies [3-5]. The most general quantum measurement is represented by a POVM. A SIC-POVM (see the definition in Sec. 2) is a POVM for which the statistics of the measurement allows the reconstruction of the quantum state. The notion of MUBs (see the definition in Sec. 3), implicit or explicit in the seminal works of Refs. [12-15], has been the object of numerous mathematical and physical investigations during the last two decades in connection with the so-called complementary observables.

The aim of this note, is to develop a unified approach to SIC-POVMs and MUBs. Our approach is based on the Wigner-Racah algebra of the chain $\mathrm{SU}(2) \supset \mathrm{U}(1)$ recently used for a study of entanglement of rotationally invariant spin systems [22] and for an angular momentum study of MUBs [20, 21].

Most of the notations in this work are standard. Let us simply mention that $\mathbb{I}$ denotes the identity opertor, the bar indicates complex conjugation, $x = [a](1)[b]$ means that x takes the values a to b by increments of 1 unit, $\delta_{a,b}$ stands for the Kronecker symbol for a and b, and $\Delta(a, b, c)$ is 1 or 0 according to as a, b and c satisfy or not the triangular inequality.

2 SIC-POVMs

A discrete SIC-POVM is a set $\{P_x : x = [1](1)[d^2]\}$ of d^2 nonnegative operators P_x acting on the Hilbert space $\mathbb{C}^d$, endowed with an inner product denoted as $\langle \, | \, \rangle$, such that

the trace relation

$$\mathrm{Tr}\,(P_x P_y) = \frac{1}{d+1}, \quad x \neq y \tag{1}$$

and the decomposition of the identity

$$\frac{1}{d}\sum_{x=1}^{d^2} P_x = \mathbb{I} \tag{2}$$

be satisfied. Therefore, by putting

$$P_x = |\Phi_x\rangle\langle\Phi_x| \tag{3}$$

with $|\phi_x\rangle \in \mathbb{C}^d$, to find d^2 operators P_x is equivalent to find d^2 vectors $|\phi_x\rangle$ in $\mathbb{C}^d$ with the property

$$|\langle\Phi_x|\Phi_y\rangle|^2 = \frac{1}{d+1}\left(d\delta_{x,y}+1\right) \tag{4}$$

The cornerstone of our approach is to identify $\mathbb{C}^d$ with a subspace $\varepsilon(j)$ of constant angular momentum $j = (d-1)/2$. Such a subspace is spanned by the set $\{|j,m\rangle : m = [-j](1)[j]\}$, where $|j,m\rangle$ is an eigenvector of the square and the z-component of a generalized angular momentum operator. Let $\mathbf{u}^{(k)}$ be the Racah unit tensor [23] of order k (with $k = [0](1)[2j]$) defined by its $2k+1$ components $u_q^{(k)}$ (where $q = [-k](1)[k]$) through

$$u_q^{(k)} = \sum_{m=-j}^{j}\sum_{m'=-j}^{j}(-1)^{j-m}\begin{pmatrix} j & k & j \\ -m & q & m' \end{pmatrix}|j,m\rangle\langle j,m'| \tag{5}$$

where $(\cdots)$ denotes a 3–jm Wigner symbol. For fixed j, the $(2j+1)^2$ operators $u_q^{(k)}$ (with $k = [0](1)[2j]$ and $q = [-k](1)[k]$) form a basis of the Hilbert space $\mathbb{C}^N$ of dimension $N = (2j+1)^2$ and consisting of operators acting on $\varepsilon(j) \sim \mathbb{C}^d$, the inner product in $\mathbb{C}^N$ being the Hilbert-Schmidt product. The formulas (involving unit tensors, 3–jm and 6–j symbols) relevant for this work are given in Appendix (see also Refs. [23-25]).

Each operator P_x can be developed as a linear combination of the operators $u_q^{(k)}$. Therefore, we can write

$$P_x = \sum_{k=0}^{2j} \sum_{q=-k}^{k} c_{kq}(x) u_q^{(k)} \tag{6}$$

where the unknown expansion coefficients $c_{kq}(x)$ are *a priori* complex numbers. The determination of the operators P_x is thus equivalent to the determination of the coefficients $c_{kq}(x)$. The latter coefficients satisfy some properties which can be summed up by the following results.

Result 1. *The coefficients $c_{kq}(x)$ of the development (6) are formally given by*

$$c_{kq}(x) = (2k+1)\overline{\langle \Phi_x | u_q^{(k)} | \Phi_x \rangle} \tag{7}$$

for $q = [-k](1)[k]$, $k = [0](1)[2j]$ and $x = [1](1)[(2j+1)^2]$.

Proof: Multiply each member of Eq. (6) by the adjoint of $u_p^{(\ell)}$ and use Eq. (39) of Appendix.

By defining the vector

$$v(x) = (v_1(x), v_2(x), \cdots, v_N(x)), \quad N = (2j+1)^2 \tag{8}$$

via

$$v_i(x) = \frac{1}{\sqrt{2k+1}} c_{kq}(x), \quad i = k^2 + k + q + 1 \tag{9}$$

the search for the set $\{P_x : x = [1](1)[N]\}$ is equivalent to the search for the set $\{v_x : x = [1](1)[N]\}$ in a second Hilbert space $\mathbb{C}^N$, with an inner product of the type $v \cdot w = \sum_{i=1}^{N} \overline{v_i}\, w_i$.

Result 2. *The first component $v_1(x)$ of $v(x)$ does not depend on x since*

$$c_{00}(x) = \frac{1}{\sqrt{2j+1}} \tag{10}$$

for $x = [1](1)[(2j+1)^2]$.

Proof: Take the trace of Eq. (6) and use Eq. (41) of Appendix.

Result 3. *The components $v_i(x)$ of $v(x)$ satisfy the complex conjugation property described by*

$$\overline{c_{kq}(x)} = (-1)^q c_{k-q}(x) \tag{11}$$

for $q = [-k](1)[k]$, $k = [0](1)[2j]$ and $x = [1](1)[(2j+1)^2]$.

Proof: Use the Hermitean property of P_x and Eq. (36) of Appendix.

Result 4. *The vectors $v(x)$ satisfy the relation*

$$v(x) \cdot v(y) = \frac{1}{2(j+1)} \left[(2j+1)\delta_{x,y} + 1\right] \tag{12}$$

for $x, y = [1](1)[(2j+1)^2]$. In detail, Eq. (12) reads

$$\sum_{k=0}^{2j} \frac{1}{2k+1} \sum_{q=-k}^{k} \overline{c_{kq}(x)} c_{kq}(y) = \frac{1}{2(j+1)} \left[(2j+1)\delta_{x,y} + 1\right] \tag{13}$$

where the sum over q is rotationally invariant.

Proof: It is sufficient to develop $\langle \phi_x | P_y | \phi_x \rangle$.

Result 5. *The coefficients $c_{kq}(x)$ are solutions of the nonlinear system given by*

$$\begin{aligned} \frac{1}{2K+1} c_{KQ}(x) &= (-1)^{2j-Q} \sum_{k=0}^{2j} \sum_{\ell=0}^{2j} \sum_{q=-k}^{k} \sum_{p=-\ell}^{\ell} \begin{pmatrix} k & \ell & K \\ -q & -p & Q \end{pmatrix} \\ &\times \overline{W} \begin{pmatrix} k & \ell & K \\ j & j & j \end{pmatrix} c_{kq}(x) c_{\ell p}(x) \end{aligned} \tag{14}$$

for $Q = [-K](1)[K]$, $K = [0](1)[2j]$ and $x = [1](1)[(2j+1)^2]$.

Proof: Consider $P_x^2 = P_x$ and use the coupling relation (44) of Appendix involving a 3–jm Wigner symbol and a $\overline{W}$ Racah coefficient (or 6–j Wigner symbol).

As a corollary of Result 5, by taking $K = 0$ and using Eqs. (40) and (43) of Appendix, we get $\|v(x)\|^2 = v(x) \cdot v(x) = 1$, a result that also follows as a particular case of Eq. (13) of Result 4.

Result 6. *All coefficients $c_{kq}(x)$ are connected through the sum rule*

$$\sum_{x=1}^{(2j+1)^2} \sum_{k=0}^{2j} \sum_{q=-k}^{k} c_{kq}(x) \begin{pmatrix} j & k & j \\ -m & q & m' \end{pmatrix} = (-1)^{j-m} (2j+1) \delta_{m,m'} \tag{15}$$

which turns out to be useful for global checking purposes.

Proof: Take the jm–jm' matrix element of the resolution of the identity in terms of the operators $P_x/(2j+1)$.

3 MUBs

A complete set of MUBs in the Hilbert space $\mathbb{C}^d$ is a set of $d(d+1)$ vectors $|a\alpha\rangle \in \mathbb{C}^d$ such that

$$|\langle a\alpha|b\beta\rangle|^2 = \delta_{\alpha,\beta}\delta_{a,b} + \frac{1}{d}(1-\delta_{a,b}) \tag{16}$$

where $a = [0](1)[d]$ and $\alpha = [0](1)[d-1]$. The indices of type a refer to the bases and, for fixed a, the index α refers to one of the d vectors of the basis corresponding to a. We know that such a complete set exists if d is a prime or the power of a prime (***e.g.***, see Refs. [13-19]).

The approach developed in Sec. 2 for SIC-POVMs can be applied to MUBs too. Let us suppose that it is possible to find $d+1$ sets S_a (with $a = [0](1)[d]$) of vectors in $\mathbb{C}^d$, each set $S_a = \{|a\alpha\rangle : \alpha = [0](1)[d-1]\}$ containing d vectors $|a\alpha\rangle$ such that Eq. (16) be satisfied. This amounts to find $d(d+1)$ operators

$$\Pi_{a\alpha} = |a\alpha\rangle\langle a\alpha| \tag{17}$$

satisfying

$$\frac{1}{d+1}\sum_{a=0}^{d}\sum_{\alpha=0}^{d-1}\Pi_{a\alpha} = \mathbb{I} \tag{18}$$

and

$$\mathrm{Tr}\,(\Pi_{a\alpha}\Pi_{b\beta}) = \delta_{\alpha,\beta}\delta_{a,b} + \frac{1}{d}(1-\delta_{a,b}) \tag{19}$$

where the trace is taken on $\mathbb{C}^d$. As in Sec. 2, we assume that the Hilbert space $\mathbb{C}^d$ is realized by a subspace $\varepsilon(j)$ of constant angular momentum $j = (d-1)/2$. Then, each

operator $\Pi_{a\alpha}$ can be developed on the basis of the $(2j+1)^2$ operators $u_q^{(k)}$ as

$$\Pi_{a\alpha} = \sum_{k=0}^{2j} \sum_{q=-k}^{k} d_{kq}(a\alpha) u_q^{(k)} \tag{20}$$

to be compared with Eq. (6). For a and α fixed, the complex coefficients $d_{kq}(a\alpha)$ define a vector

$$w(a\alpha) = (w_1(a\alpha), w_2(a\alpha), \cdots, w_N(a\alpha)), \quad N = (2j+1)^2 \tag{21}$$

in the Hilbert space $\mathbb{C}^N$. The components $w_i(a\alpha)$ are given by

$$w_i(a\alpha) = \frac{1}{\sqrt{2k+1}} d_{kq}(a\alpha), \quad i = k^2 + k + q + 1 \tag{22}$$

Results 1 to 6 can be transcribed in terms of $d_{kq}(a\alpha)$ or $w(a\alpha)$. This leads to Results 7 to 12 given below without proof.

Result 7. *We have the formal expression*

$$d_{kq}(a\alpha) = (2k+1)\overline{\langle a\alpha | u_q^{(k)} | a\alpha \rangle} \tag{23}$$

for $q = [-k](1)[k]$, $k = [0](1)[2j]$, $a = [0](1)[2j+1]$ *and* $\alpha = [0](1)[2j]$.

Result 8. *We have the formula*

$$d_{00}(a\alpha) = \frac{1}{\sqrt{2j+1}} \tag{24}$$

for $a = [0](1)[2j+1]$ *and* $\alpha = [0](1)[2j]$.

Result 9. *We have the complex conjugation property*

$$\overline{d_{kq}(a\alpha)} = (-1)^q d_{k-q}(a\alpha) \tag{25}$$

for $q = [-k](1)[k]$, $k = [0](1)[2j]$, $a = [0](1)[2j+1]$ *and* $\alpha = [0](1)[2j]$.

Result 10. *We have the scalar product formula*

$$w(a\alpha) \cdot w(b\beta) = \delta_{\alpha,\beta}\delta_{a,b} + \frac{1}{2j+1}(1 - \delta_{a,b}) \tag{26}$$

for $a, b = [0](1)[2j+1]$ *and* $\alpha, \beta = [0](1)[2j]$.

Result 11. *We have the tensor product formula*

$$\begin{aligned}\frac{1}{2K+1}d_{KQ}(a\alpha) &= (-1)^{2j-Q}\sum_{k=0}^{2j}\sum_{\ell=0}^{2j}\sum_{q=-k}^{k}\sum_{p=-\ell}^{\ell}\begin{pmatrix} k & \ell & K \\ -q & -p & Q\end{pmatrix} \\ &\times \overline{W}\begin{pmatrix} k & \ell & K \\ j & j & j\end{pmatrix} d_{kq}(a\alpha)d_{\ell p}(a\alpha) \end{aligned} \tag{27}$$

for $Q = [-K](1)[K]$, $K = [0](1)[2j]$, $a = [0](1)[2j+1]$ *and* $\alpha = [0](1)[2j]$.

Result 12. *We have the sum rule*

$$\sum_{a=0}^{2j+1}\sum_{\alpha=0}^{2j}\sum_{k=0}^{2j}\sum_{q=-k}^{k} d_{kq}(a\alpha)\begin{pmatrix} j & k & j \\ -m & q & m'\end{pmatrix} = (-1)^{j-m}(2j+1)\delta_{m,m'} \tag{28}$$

which involves all coefficients $d_{kq}(a\alpha)$.

4 CONCLUSIONS

Although the structure of the relations describing Results 1 to 6, on one hand, and Results 7 to 12, on the other hand, is very similar, there are deep differences between the two sets of results. The similarities are reminiscent of the fact that both MUBs and SIC-POVMs can be linked to finite affine planes [10, 17, 18] and to complex projective 2–designs [6, 8, 16, 19]. On the other side, there are two arguments in favor of the differences between Results 4 and 10. First, the problem of constructing SIC-POVMs in dimension d is not equivalent to the existence of an affine plane of order d [10]. Second, there is a consensus around the conjecture according to which there exists a complete set of MUBs in dimension d if and only if there exists an affine plane of order d [17].

In dimension d, to find d^2 operators P_x of a SIC-POVM acting on the Hilbert space $\mathbb{C}^d$ amounts to find d^2 vectors $v(x)$ in the Hilbert space $\mathbb{C}^N$ with $N = d^2$ satisfying

$$\|v_x\| = 1, \quad v(x)\cdot v(y) = \frac{1}{d+1} \text{ for } x \neq y \tag{29}$$

(the norm $\|v(x)\|$ of each vector $v(x)$ is 1 and the angle ω_{xy} of any pair of vectors $v(x)$ and $v(y)$ is $\omega_{xy} = \cos^{-1}[1/(d+1)]$ for $x \neq y$).

In a similar way, to find $d+1$ MUBs of $\mathbb{C}^d$ is equivalent to find $d+1$ sets of d vectors, *i.e.*, $d(d+1)$ vectors in all, $w(a\alpha)$ in $\mathbb{C}^N$ with $N=d^2$ satisfying

$$w(a\alpha)\cdot w(a\beta)=\delta_{\alpha,\beta},\quad w(a\alpha)\cdot w(b\beta)=\frac{1}{d}\text{ for }a\neq b \tag{30}$$

(each set S_a consists of d orthonormalized vectors and the angle $\omega_{a\alpha b\beta}$ of any vector $w(a\alpha)$ of a set S_a with any vector $w(b\beta)$ of a set S_b is $\omega_{a\alpha b\beta}=\cos^{-1}(1/d)$ for $a\neq b$).

According to a well accepted conjecture [6, 8], SIC-POVMs should exist in any dimension. The present study shows that in order to prove this conjecture it is sufficient to prove that Eq. (29) admit solutions for any value of d.

The situation is different for MUBs. In dimension d, it is known that there exist $d+1$ sets of d vectors of type $|a\alpha\rangle$ in $\mathbb{C}^d$ satisfying Eq. (16) when d is a prime or the power of a prime. This shows that Eq. (30) can be solved for d prime or power of a prime. For d prime, it is possible to find an explicit solution of Eq. (16). In fact, we have [20, 21]

$$\begin{aligned}|a\alpha\rangle &= \frac{1}{\sqrt{2j+1}}\sum_{m=-j}^{j}\omega^{(j+m)(j-m+1)a/2+(j+m)\alpha}|j,m\rangle,\\ \omega &= \exp\left(\mathrm{i}\frac{2\pi}{2j+1}\right),\quad j=\frac{1}{2}(d-1)\end{aligned} \tag{31}$$

for $a=[0](1)[2j]$ and $\alpha=[0](1)[2j]$ while

$$|a\alpha\rangle=|j,m\rangle \tag{32}$$

for $a=2j+1$ and $\alpha=j+m=[0](1)[2j]$. Then, Eq. (23) yields

$$\begin{aligned}d_{kq}(a\alpha) &= \frac{2k+1}{2j+1}\sum_{m=-j}^{j}\sum_{m'=-j}^{j}\omega^{\theta(m,m')}(-1)^{j-m}\begin{pmatrix}j & k & j\\ -m & 0 & m'\end{pmatrix},\\ \theta(m,m') &= (m-m')\left[\frac{1}{2}(1-m-m')a+\alpha\right]\end{aligned} \tag{33}$$

for $a=[0](1)[2j]$ and $\alpha=[0](1)[2j]$ while

$$d_{kq}(a\alpha)=\delta_{q,0}(2k+1)(-1)^{j-m}\begin{pmatrix}j & k & j\\ -m & 0 & m\end{pmatrix} \tag{34}$$

for $a = 2j + 1$ and $\alpha = j + m = [0](1)[2j]$. It can be shown that Eqs. (33) and (34) are in agreement with Results 7 to 12. We thus have a solution of the equations for Results 7 to 12 when d is prime. As an open problem, it would be worthwhile to find an explicit solution for the coefficients $d_{kq}(a\alpha)$ when $d = 2j + 1$ is any positive power of a prime. Finally, note that to prove (or disprove) the conjecture according to which a complete set of MUBs in dimension d exists only if d is a prime or the power of a prime is equivalent to prove (or disprove) that Eq. (30) has a solution only if d is a prime or the power of a prime.

APPENDIX: WIGNER-RACAH ALGEBRA OF $\mathrm{SU}(2) \supset \mathrm{U}(1)$

We limit ourselves to those basic formulas for the Wigner-Racah algebra of the chain $\mathrm{SU}(2) \supset \mathrm{U}(1)$ which are necessary for the derivation of Results 1 to 12. The summations in this appendix have to be extended to the allowed values for the involved magnetic and angular momentum quantum numbers.

The definition (5) of the components $u_q^{(k)}$ of the Racah unit tensor $\mathbf{u}^{(k)}$ yields

$$\langle j, m | u_q^{(k)} | j, m' \rangle = (-1)^{j-m} \begin{pmatrix} j & k & j \\ -m & q & m' \end{pmatrix} \tag{35}$$

from which we easily obtain the Hermitean conjugation property

$$u_q^{(k)\dagger} = (-1)^q u_{-q}^{(k)} \tag{36}$$

The 3–jm Wigner symbol in Eq. (35) satisfy the orthogonality relations

$$\sum_{mm'} \begin{pmatrix} j & j' & k \\ m & m' & q \end{pmatrix} \begin{pmatrix} j & j' & \ell \\ m & m' & p \end{pmatrix} = \frac{1}{2k+1} \delta_{k,\ell} \delta_{q,p} \Delta(j, j', k) \tag{37}$$

and

$$\sum_{kq} (2k+1) \begin{pmatrix} j & j' & k \\ m & m' & q \end{pmatrix} \begin{pmatrix} j & j' & k \\ M & M' & q \end{pmatrix} = \delta_{m,M} \delta_{m',M'} \tag{38}$$

The trace relation on the space $\varepsilon(j)$

$$\mathrm{Tr}\left(u_q^{(k)\dagger}u_p^{(\ell)}\right)=\frac{1}{2k+1}\delta_{k,\ell}\delta_{q,p}\Delta(j,j,k) \tag{39}$$

easily follows by combining Eqs. (35) and (37). Furthermore, by introducing

$$\begin{pmatrix} j & j' & 0 \\ m & -m' & 0 \end{pmatrix}=\delta_{j,j'}\delta_{m,m'}(-1)^{j-m}\frac{1}{\sqrt{2j+1}} \tag{40}$$

in Eq. (37), we obtain the sum rule

$$\sum_m(-1)^{j-m}\begin{pmatrix} j & k & j \\ -m & q & m \end{pmatrix}=\sqrt{2j+1}\delta_{k,0}\delta_{q,0}\Delta(j,k,j) \tag{41}$$

known in spectroscopy as the barycenter theorem.

There are several relations involving 3–jm symbols and $\overline{W}$ coefficients. In particular, we have

$$\begin{aligned}\sum_{mm'M}(-1)^{j-M}\begin{pmatrix} j & k & j \\ -m & q & M \end{pmatrix}\begin{pmatrix} j & \ell & j \\ -M & p & m' \end{pmatrix}\\ \times\begin{pmatrix} j & K & j \\ -m & Q & m' \end{pmatrix}=(-1)^{2j-Q}\begin{pmatrix} k & \ell & K \\ -q & -p & Q \end{pmatrix}\overline{W}\begin{pmatrix} k & \ell & K \\ j & j & j \end{pmatrix}\end{aligned} \tag{42}$$

Note that the introduction of

$$\overline{W}\begin{pmatrix} k & \ell & 0 \\ j & j & J \end{pmatrix}=\delta_{k,\ell}(-1)^{j+k+J}\frac{1}{\sqrt{(2k+1)(2j+1)}} \tag{43}$$

in Eq. (42) gives back Eq. (37). Equation (42) is central in the derivation of the coupling relation

$$u_q^{(k)}u_p^{(\ell)}=\sum_{KQ}(-1)^{2j-Q}(2K+1)\begin{pmatrix} k & \ell & K \\ -q & -p & Q \end{pmatrix}\overline{W}\begin{pmatrix} k & \ell & K \\ j & j & j \end{pmatrix}u_Q^{(K)} \tag{44}$$

Equation (44) makes it possible to calculate the commutator $[u_q^{(k)},u_p^{(\ell)}]$ which shows that the set $\{u_q^{(k)}:k=[0](1)[2j],q=[-k](1)[k]\}$ can be used to span the Lie algebra of the unitary group U($2j+1$). The latter result is at the root of the expansions (6) and (20).

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