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A SU(2) recipe for mutually unbiased bases

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A SU(2) recipe for mutually unbiased bases

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Abstract

A simple recipe for generating a complete set of mutually unbiased bases in dimension $2j + 1$, with $2j$ integer and $2j + 1$ prime, is developed from a single matrix V_a acting on a space of constant angular momentum j and defined in terms of the irreducible characters of the cyclic group C_{2j+1} . This recipe yields an (apparently new) compact formula for the vectors spanning the various mutually unbiased bases. In dimension $(2j + 1)^e$, with $2j$ integer, $2j + 1$ prime and e positive integer, the use of direct products of matrices of type V_a makes it possible to generate mutually unbiased bases. As two pending results, the matrix V_a is used in the derivation of a polar decomposition of SU(2) and of a FFZ algebra.

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1 Introduction

The notion of mutually unbiased bases (MUBs),^{1–23} originally introduced by Schwinger and named by Wootters,³ is of paramount importance in quantum information theory, especially in quantum cryptography and quantum state tomography. Let us recall that two orthonormal bases $\{|an_\alpha\rangle : n_\alpha = 0, 1, \dots, d-1\}$ and $\{|bn_\beta\rangle : n_\beta = 0, 1, \dots, d-1\}$ of a d -dimensional Hilbert space, with an inner product denoted as $\langle | \rangle$, are said to be mutually unbiased if and only if

$$|\langle an_\alpha | bn_\beta \rangle| = \delta(a, b)\delta(n_\alpha, n_\beta) + [1 - \delta(a, b)]\frac{1}{\sqrt{d}}.$$

In dimension d , the maximum number of pairwise MUBs is $d+1$;^{1–5} a set consisting of $d+1$ pairwise MUBs is called a complete set. As a matter of fact, the upper bound $d+1$ is attained when d is a prime number or the power of a prime number.^{2–9,16} There are numerous ways for constructing complete sets of MUBs,^{1–23} most of them being based on discrete Fourier analysis in Galois fields and Galois rings,^{3,9,12,14,16,19,21} discrete Wigner functions,^{3,10,21,22} generalized Pauli matrices.^{5–8,10} Note also that the existence of MUBs can be related to the problem of finding mutually orthogonal Latin squares^{11,15,22} and a solution of the mean King problem.^{11,22} Let us also mention that the existence of MUBs has been addressed by various authors from the point of view of finite geometries.^{13,15,17,19} Finally, Lie algebra approaches to MUBs have been developed recently.^{20,23}

The main aim of this note is to give a simple algorithm for generating MUBs in dimension d where d is a prime number. The case where d is the power of a prime number is briefly examined. The present work constitutes a continuation of the ones in Ref. 23.

2 The Main Results

Let $\epsilon(j)$ be a $(2j+1)$ -dimensional Hilbert space of constant angular momentum j (the quantum number j is such that $2j \in \mathbf{N}^*$). An orthonormal basis for $\epsilon(j)$ is provided by the set $\{|j, m\rangle : m = j, j-1, \dots, -j\}$ where the angular momentum state vectors $|j, m\rangle$, sometimes referred to as spherical or computational or Fock states, are eigenstates of the square J^2 of a generalized angular momentum and its z -component J_z .

Following the suggestion made in Ref. 23 of “redefining the operator U_r ” used in a study of $SU(2)$, we introduce the $(2j+1)$ -dimensional unitary matrix

$$V_a = \begin{pmatrix} 0 & q^a & 0 & \cdots & 0 \\ 0 & 0 & q^{2a} & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & q^{2ja} \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad a \in \{0, 1, \dots, 2j\},$$

built on the spherical or standard basis $b_s = (|j, j\rangle, |j, j-1\rangle, \dots, |j, -j\rangle)$. Here, the parameter q is a root of unity defined by

$$q = \exp\left(i\frac{2\pi}{2j+1}\right).$$

We have the immediate property

$$\text{Tr} (V_a^\dagger V_b) = (2j + 1)\delta(a, b).$$

The matrix V_a is a generalization of the matrix U_r with $r \in \mathbf{R}$ considered in Ref. 24 in the framework of a polar decomposition of $\text{SU}(2)$ and used in Ref. 23 for generating MUBs in the cases $d = 2$ and 3. The set $\{V_0, V_1, \dots, V_{2j}\}$ of the $2j + 1$ matrices V_a is constructed from the $2j + 1$ irreducible character vectors of the cyclic group C_{2j+1} . Indeed, the nonzero matrix elements of the matrix V_a are given by the irreducible character vector

$$\chi^a = (1, q^a, \dots, q^{2ja})$$

of C_{2j+1} .

It is straightforward to find the eigenvalues and eigenvectors of V_a . As a result, the spectrum of V_a is non-degenerate. The eigenvector $|jan_\alpha\rangle$ corresponding to the eigenvalue

$$\lambda(jan_\alpha) = q^{ja-n_\alpha}$$

reads

$$|jan_\alpha\rangle = \frac{1}{\sqrt{2j+1}} \sum_{m=-j}^j q^{\frac{1}{2}(j+m)(j-m+1)a+(j+m)n_\alpha} |j, m\rangle, \quad (1)$$

where $n_\alpha = 0, 1, \dots, 2j$. The $2j + 1$ eigenvectors $|jan_\alpha\rangle$ of the matrix V_a generate an orthonormal basis b_a of the space $\epsilon(j)$. For fixed a , the bases b_a and b_s are mutually unbiased. More specifically, we have the following result.

Result 1. In the case where $2j + 1$ is a prime integer, the set comprising the spherical basis b_s and the $2j + 1$ bases b_a for $a = 0, 1, \dots, 2j$ constitute a complete set of $2(j + 1)$ MUBs.

At this point, a natural question arises. How to construct a complete set of MUBs for the direct product space $\epsilon(j) \otimes \epsilon(j) \otimes \dots \otimes \epsilon(j)$ (with e factors) of dimension $d = (2j+1)^e$, where $2j + 1$ is prime and e is an integer greater or equal to 2? The answer follows from the following result.

Result 2. In the case where $2j + 1$ is a prime integer, the eigenvectors of the matrices

$$W_{a_1 a_2 \dots a_e} = V_{a_1} \otimes V_{a_2} \otimes \dots \otimes V_{a_e}, \quad a_i \in \{0, 1, \dots, 2j\}, \quad i = 1, 2, \dots, e,$$

together with the d -dimensional computational basis can be arranged to form a complete set of $d + 1 = (2j + 1)^e + 1$ MUBs.

The proofs of Results 1 and 2 can be obtained from an adaptation of the proofs in Refs. 5-7, 12 and 21. The term ‘‘arranged’’ in Result 2 means that auxilliary matrices need to be introduced in order to deal with the degeneracy problem.

As a corollary of Result 1, we obtain the sum rule

$$\left| \sum_{k=0}^{d-1} q^{\frac{1}{2}k(d-k)(a-b)+k(n_\alpha-n_\beta)} \right| = d\delta(a, b)\delta(n_\alpha, n_\beta) + \sqrt{d}[1 - \delta(a, b)],$$

with

$$q = \exp\left(i\frac{2\pi}{d}\right), \quad a, b \in \{0, 1, \dots, d-1\}, \quad n_\alpha, n_\beta \in \{0, 1, \dots, d-1\},$$

where d is a prime number.

3 Two Related Results

We would like to outline two Lie-like aspects of our approach.

First, we can find a polar decomposition of the shift operators j_+ and j_- of the Lie group $SU(2)$ in terms of the unitary operator v_a associated to the matrix V_a . The operator v_a satisfies

$$v_a|j, m\rangle = q^{(j-m)a}[1 - \delta(m, j)]|j, m+1\rangle + \delta(m, j)|j, -j\rangle$$

for $m = j, j-1, \dots, -j$. Following Refs. 23 and 24, let us define the Hermitean operator h through

$$h|j, m\rangle = \sqrt{(j+m)(j-m+1)}|j, m\rangle.$$

We can show that the linear operators

$$j_+ = hv_a, \quad j_- = v_a^\dagger h, \quad j_z = \frac{1}{2}(h^2 - v_a^\dagger h^2 v_a)$$

have the following action

$$j_\pm|j, m\rangle = q^{\pm(j\mp m + \frac{1}{2}\mp\frac{1}{2})a}\sqrt{(j-m)(j+m+1)}|j, m\pm 1\rangle, \quad j_z|j, m\rangle = m|j, m\rangle \quad (2)$$

on the standard state vector $|j, m\rangle$ for $m = j, j-1, \dots, -j$. As a consequence, we get

$$[j_z, j_\pm] = \pm j_\pm, \quad [j_+, j_-] = 2j_z.$$

Hence, the operators j_+ , j_- and j_z span the Lie algebra of $SU(2)$. This result is to be compared with similar results obtained in Refs. 21 and 23-25 without the occurrence of the parameter a . It is to be emphasized that this result holds for any value of a ($a = 0, 1, \dots, 2j$). However, note that the action of j_\pm on $|j, m\rangle$ depends on a . The Condon and Shortley phase convention used in atomic spectroscopy amounts to take $a = 0$ in Eq. (2).

Second, the cyclic character of the irreducible representations of C_{2j+1} renders possible to express V_a in function of V_0 . In fact, we have

$$V_a = V_0 Z^a,$$

where

$$Z = \text{diag}(1, q, \dots, q^{2j}).$$

The matrices V_a and Z have an interesting property, namely, they q -commute in the sense that

$$V_a Z - q Z V_a = 0.$$

By defining

$$T_m = q^{\frac{1}{2}m_1m_2} V_a^{m_1} Z^{m_2}, \quad m = (m_1, m_2) \in \mathbf{N}^{*2},$$

we easily obtain the commutator

$$[T_m, T_n] = 2i \sin \left(\frac{\pi}{2j+1} m \wedge n \right) T_{m+n},$$

where

$$m \wedge n = m_1n_2 - m_2n_1, \quad m + n = (m_1 + n_1, m_2 + n_2),$$

so that the linear operators T_m span the FFZ infinite dimensional Lie algebra introduced by Fairlie, Fletcher and Zachos.²⁶ The latter result parallels the ones obtained, on one hand, from a study of k -fermions and of the Dirac quantum phase operator through a q -deformation of the harmonic oscillator²⁷ and, on the other hand, from an investigation of correlation measure for finite quantum systems.²⁵

4 Closing Remarks

In the case where $d = 2j + 1$ is a prime number, Result 1 provides us with a simple mean for generating a complete set of $d + 1$ MUBs from the knowledge of a single matrix, viz., the matrix V_a . It should be noted that when $2j + 1$ is not a prime number, Eq. (1) can be used for spanning MUBs as well; however, in that case, it is not possible to generate a complete set of MUBs.

The main interest of our approach relies on the fact that MUBs can be constructed from a simple generic matrix V_a and yields calculations easily codable on a computer. In addition, the matrix V_a turns out to be of physical interest and plays an important role in the polar decomposition of $SU(2)$ and for the derivation of the FFZ algebra.

These matters, inherited from a q -deformation approach to symmetry and supersymmetry,^{27,28} will be developed in a forthcoming paper in a larger context involving MUBs, useful in quantum information, and symmetry adapted bases, useful in molecular physics and quantum chemistry.

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