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## **A SU(2) recipe for mutually unbiased bases**

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# A $SU(2)$ recipe for mutually unbiased bases

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## Abstract

A simple recipe for generating a complete set of mutually unbiased bases in dimension  $(2j + 1)^e$ , with  $2j + 1$  prime and  $e$  positive integer, is developed from a single matrix acting on a space of constant angular momentum  $j$  and defined in terms of the irreducible characters of the cyclic group  $C_{2j+1}$ . As two pending results, this matrix is used in the derivation of a polar decomposition of  $SU(2)$  and of a FFZ algebra.

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## Introduction

The notion of mutually unbiased bases (MUBs) [1-23], originally introduced by Schwinger and named by Wootters [3], is of paramount importance in quantum information theory, especially in quantum cryptography and quantum state tomography. Let us recall that two orthonormal bases  $\{|an_\alpha\rangle : n_\alpha = 0, 1, \dots, d-1\}$  and  $\{|bn_\beta\rangle : n_\beta = 0, 1, \dots, d-1\}$  of a  $d$ -dimensional Hilbert space, with an inner product denoted as  $\langle | \rangle$ , are said to be mutually unbiased if and only if

$$|\langle an_\alpha | bn_\beta \rangle| = \delta(a, b)\delta(n_\alpha, n_\beta) + [1 - \delta(a, b)]\frac{1}{\sqrt{d}}$$

In dimension  $d$ , the maximum number of pairwise MUBs is  $d+1$  [1-5]; a set consisting of  $d+1$  pairwise MUBs is called a complete set. As a matter of fact, the upper bound  $d+1$  is attained when  $d$  is a prime number or the power of a prime number [2-9,16]. There are numerous ways for constructing complete sets of MUBs [1-23], most of them being based on discrete Fourier analysis in Galois fields and Galois rings [3,9,12,14,16,19,21], discrete Wigner functions [3,10,21,22], generalized Pauli matrices [5-8,10]. Note also that the existence of MUBs can be related to the problem of finding mutually orthogonal Latin squares [11,15,22] and a solution of the mean King problem [11,22]. Let us also mention that the existence of MUBs has been addressed by various authors from the point of view of finite geometries [13,15,17,19]. Finally, Lie algebra approaches to MUBs have been developed recently [20,23].

The aim of this letter is to give a simple algorithm for generating MUBs in dimension  $d$  where  $d$  is a prime number or, more generally, the power of a prime number.

## The main results

Let  $\epsilon(j)$  be a  $(2j+1)$ -dimensional Hilbert space of constant angular momentum  $j$  (the quantum number  $j$  is such that  $2j \in \mathbf{N}^*$ ). An orthonormal basis for  $\epsilon(j)$  is provided by the set  $\{|j, m\rangle : m = j, j-1, \dots, -j\}$  where the angular momentum state vectors  $|j, m\rangle$ , sometimes referred to as spherical or computational or Fock states, are eigenstates of the square  $J^2$  of a generalized angular momentum and its  $z$ -component  $j_z$ .

We now consider the  $(2j+1)$ -dimensional unitary matrix

$$V_a = \begin{pmatrix} 0 & q^a & 0 & \dots & 0 \\ 0 & 0 & q^{2a} & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & q^{2ja} \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}, \quad a \in \{0, 1, \dots, 2j\}$$

built on the spherical or standard basis  $b_s = (|j, j\rangle, |j, j-1\rangle, \dots, |j, -j\rangle)$ . Here, the parameter  $q$  is a root of unity defined by

$$q = \exp\left(i\frac{2\pi}{2j+1}\right)$$

We have the immediate property

$$\text{Tr}(V_a^\dagger V_b) = (2j+1)\delta(a,b)$$

The matrix  $V_a$  is a generalization of the matrix  $U_r$  with  $r \in \mathbf{R}$  introduced in [24] in the framework of a polar decomposition of  $\text{SU}(2)$  and used in [23] for generating MUBs in the cases  $d = 2$  and  $3$ . The set  $\{V_0, V_1, \dots, V_{2j}\}$  of the  $2j+1$  matrices  $V_a$  is constructed from the  $2j+1$  irreducible character vectors of the cyclic group  $C_{2j+1}$ . Indeed, the nonzero matrix elements of the matrix  $V_a$  are given by the irreducible character vector

$$\chi^a = (1, q^a, \dots, q^{2ja})$$

of  $C_{2j+1}$ .

It is straightforward to find the eigenvalues and eigenvectors of  $V_a$ . As a result, the spectrum of  $V_a$  is non-degenerate. The eigenvector  $|jn_\alpha a\rangle$  corresponding to the eigenvalue

$$\lambda(jn_\alpha) = q^{ja+n_\alpha}$$

reads

$$|jan_\alpha\rangle = \frac{1}{\sqrt{2j+1}} \sum_{m=-j}^j q^{\frac{1}{2}(j+m)(j-m+1)a+(j+m)n_\alpha} |j, m\rangle \quad (1)$$

The  $2j+1$  eigenvectors  $|jan_\alpha\rangle$  of the matrix  $V_a$  generate an orthonormal basis  $b_a$  of the space  $\epsilon(j)$ . For fixed  $a$ , the bases  $b_a$  and  $b_s$  are mutually unbiased. More specifically, we have the following result.

**Result 1.** In the case where  $2j+1$  is a prime integer, the set comprising the spherical basis  $b_s$  and the  $2j+1$  bases  $b_a$  for  $a = 0, 1, \dots, 2j$  constitute a complete set of  $2(j+1)$  MUBs.

At this point, a natural question arises. How to construct a complete set of MUBs for the direct product space  $\epsilon(j) \otimes \epsilon(j) \otimes \dots \otimes \epsilon(j)$  (with  $e$  factors) of dimension  $d = (2j+1)^e$ , where  $2j+1$  is prime and  $e$  is an integer greater or equal to 2? The answer follows from the following result.

**Result 2.** In the case where  $2j+1$  is a prime integer, the eigenvectors of the matrices

$$W_{a_1 a_2 \dots a_e} = V_{a_1} \otimes V_{a_2} \otimes \dots \otimes V_{a_e}, \quad a_i \in \{0, 1, \dots, 2j\}, \quad i = 1, 2, \dots, e$$

together with the  $d$ -dimensional computational basis can be arranged to form a complete set of  $d+1 = (2j+1)^e + 1$  MUBs.

The proofs of Results 1 and 2 can be obtained from an adaptation of the proofs in Refs. [5-7,12,21].

## Two related results

We would like to outline two Lie-like aspects of this work.

First, we can find a polar decomposition of the shift operators  $j_+$  and  $j_-$  of the Lie group  $SU(2)$  in terms of the unitary operator  $v_a$  associated to the matrix  $V_a$ . The operator  $v_a$  satisfies

$$v_a|j, m\rangle = q^{(j-m)a}[1 - \delta(m, j)]|j, m+1\rangle + \delta(m, j)|j, -j\rangle$$

for  $m = j, j-1, \dots, -j$ . Following [23,24], let us define the Hermitean operator  $h$  through

$$h|j, m\rangle = \sqrt{(j+m)(j-m+1)}|j, m\rangle$$

We can show that the linear operators

$$j_+ = hv_a, \quad j_- = v_a^\dagger h, \quad j_z = \frac{1}{2}(h^2 - v_a^\dagger h^2 v_a)$$

have the following action

$$j_\pm|j, m\rangle = q^{\pm(j\mp m + \frac{1}{2}\mp\frac{1}{2})a}\sqrt{(j-m)(j+m+1)}|j, m\pm 1\rangle, \quad j_z|j, m\rangle = m|j, m\rangle \quad (2)$$

on the standard state vector  $|j, m\rangle$  for  $m = j, j-1, \dots, -j$ . As a consequence, we get

$$[j_z, j_\pm] = \pm j_\pm, \quad [j_+, j_-] = 2j_z$$

Hence, the operators  $j_+$ ,  $j_-$  and  $j_z$  span the Lie algebra of  $SU(2)$ . This result is to be compared with similar results obtained in [21,23-25] without the occurrence of the parameter  $a$ . It is to be emphasised that this result holds for any value of  $a$  ( $a = 0, 1, \dots, 2j$ ). However, note that the action of  $j_\pm$  on  $|j, m\rangle$  depends on  $a$ . The Condon and Shortley phase convention used in atomic spectroscopy amounts to take  $a = 0$  in Eq. (2).

Second, the cyclic character of the irreducible representations of  $C_{2j+1}$  render possible to express  $V_a$  in function of  $V_0$ . In fact, we have

$$V_a = V_0 Z^a$$

where

$$Z = \text{diag}(1, q, \dots, q^{2j})$$

The matrices  $V_a$  and  $Z$  have an interesting property, namely, they  $q$ -commute in the sense that

$$V_a Z - q Z V_a = 0$$

By defining

$$T_m = q^{\frac{1}{2}m_1 m_2} V_a^{m_1} Z^{m_2}, \quad m = (m_1, m_2) \in \mathbf{N}^{*2}$$

we easily obtain the commutator

$$[T_m, T_n] = 2i \sin\left(\frac{\pi}{2j+1} m \wedge n\right) T_{m+n}$$

where

$$m \wedge n = m_1 n_2 - m_2 n_1, \quad m + n = (m_1 + n_1, m_2 + n_2)$$

so that the linear operators  $T_m$  span the FFZ infinite dimensional Lie algebra introduced by Fairlie, Fletcher and Zachos [26]. The latter result parallels the ones obtained, on one hand, from a study of  $k$ -fermions and of the Dirac quantum phase operator through a  $q$ -deformation of the harmonic oscillator [27] and, on the other hand, from an investigation of correlation measure for finite quantum systems [25].

## Closing remarks

In the case where  $d$  is a prime number  $(2j + 1)$  or the power of a prime number  $((2j + 1)^e)$ , Results 1 and 2 provide us with a simple mean for generating a complete set of MUBs from the knowledge of a single matrix, viz., the matrix  $V_a$ . It should be noted that when  $2j + 1$  is not a prime number, Eq. (1) can be used for spanning MUBs as well; however, in that case, it is not possible to generate a complete set of MUBs.

The main interest of our approach relies on the fact that MUBs can be constructed from a simple generic matrix  $V_a$  and yields calculations easily codable on a computer. In addition, the matrix  $V_a$  turns out to be of physical interest and plays an important role in the polar decomposition of  $SU(2)$  and for the derivation of the FFZ algebra.

These matters, inherited from a  $q$ -deformation approach to symmetry and supersymmetry [27,28], will be developed in a full paper.

## References

- [1] P. Delsarte, J.M. Goethals and J.J. Seidel, *Philips Res. Repts.* **30**, 91 (1975).
- [2] I.D. Ivanović, *J. Phys.* **A14**, 3241 (1981).
- [3] W.K. Wootters, *Ann. Phys.* **176**, 1 (1987). W.K. Wootters and B.D. Fields, *Ann. Phys.* **191**, 363 (1989). K.S. Gibbons, M.J. Hoffman and W.K. Wootters, *Phys. Rev.* **A70**, 062101 (2004). W.K. Wootters, *IBM J. Res. Dev.* **48**, No. 1 (2004).
- [4] A.R. Calderbank, P.J. Cameron, W.M. Kantor and J.J. Seidel, *Proc. London Math. Soc.* **75**, 436 (1997).
- [5] S. Bandyopadhyay, P.O. Boykin, V. Roychowdhury and F. Vatan, *Algorithmica* **34**, 512 (2002).
- [6] J. Lawrence, Č. Brukner and A. Zeilinger, *Phys. Rev.* **A65**, 032320 (2002).
- [7] S. Chaturvedi, *Phys. Rev.* **A65**, 044301 (2002).
- [8] A.O. Pittenger and M.H. Rubin, *Linear Alg. Appl.* **390**, 255 (2004); *J. Phys.* **A38**, 6005 (2005).
- [9] A. Klappenecker and M. Rötteler, *Lecture Notes in Computer Science* **2948**, 137 (2004). A. Klappenecker, M. Rötteler, I.E. Shparlinski and A. Winterhof, *J. Math. Phys.* **46**, 082104 (2005).
- [10] T. Durt, *J. Phys.* **A38**, 5267 (2005).
- [11] P. Wocjan and T. Beth, *Quant. Inf. and Comp.* **5**, 93 (2005).
- [12] A.B. Klimov, L.L. Sánchez-Soto and H. de Guise, *J. Phys.* **A38**, 2747 (2005). J.L. Romero, G. Björk, A.B. Klimov and L.L. Sánchez-Soto, *Phys. Rev.* **A72**, 062310 (2005).

- [13] M. Saniga, M. Planat and H. Rosu, *J. Opt. B: Quantum Semiclass. Opt.* **6**, L19 (2004). M. Saniga and M. Planat, *Chaos, Solitons and Fractals* **26**, 1267 (2005); *J. Phys.* **A39**, 435 (2006).
- [14] M. Planat and H. Rosu, *Eur. Phys. J.* **D36**, 133 (2005). M. Planat, *Int. J. Mod. Phys B* (to appear).
- [15] W.K. Wootters, *Found. Phys.* (to appear).
- [16] C. Archer, *J. Math. Phys.* **46**, 022106 (2005).
- [17] I. Bengtsson, quant-ph/0406174. I. Bengtsson and Å. Ericsson, *Open Sys. & Information Dyn.* **12**, 107 (2005).
- [18] M. Grassl, in: *Proc. ERATO Conf. on Quant. Inf. Science (EQIS 2004)*, Ed. J. Gruska (Tokyo, 2005). p. 60.
- [19] M. Planat, H. Rosu, S. Perrine and M. Saniga, *Prog. Theor. Phys.* (to appear).
- [20] P.O. Boykin, M. Sitharam, P.H. Tiep and P. Wocjan, *Phys. Rev.* **A72**, 062310 (2005).
- [21] A. Vourdas, *Phys. Rev.* **A41**, 1653 (1990); *J. Phys.* **A29**, 4275 (1996); *Rep. Prog. Phys.* **67**, 267 (2004); *J. Phys.* **A38**, 8453 (2005).
- [22] L. Vaidman, Y. Aharonov and D.Z. Albert, *Phys. Rev. Lett.* **58**, 1385 (1987). Y. Aharonov and B.-G. Englert, *Z. Naturforsch.* **56a**, 16 (2001). B.-G. Englert and Y. Aharonov, *Phys. Lett.* **A284**, 1 (2001). P.K. Aravind, *Z. Naturforsch.* **58a**, 68 and 2212 (2003). A. Hayashi, M. Horibe and T. Hashimoto, *Phys. Rev.* **A71**, 052331 (2005). J.P. Paz, A.J. Roncaglia and M. Saraceno, *Phys. Rev.* **A72**, 012309 (2005).
- [23] M.R. Kibler, *Collect. Czech. Chem. Commun.* **70**, 771 (2005); *Int. J. Mod. Phys. B* (to appear).
- [24] M.R. Kibler, in: *Symmetry and Structural Properties of Condensed Matter*, Eds. T. Lulek, B. Lulek and A. Wal (World Scientific, Singapore, 1999). M. Kibler and M. Daoud, *Recent Res. Devel. Quantum. Chem.* **2**, 91 (2001).
- [25] M. Chaichian and D. Ellinas, *J. Phys.* **A23**, L291 (1990). D. Ellinas, *J. Mod. Opt.* **38**, 2393 (1991). D. Ellinas and E.G. Floratos, *J. Phys.* **A32**, L63 (1999).
- [26] D.B. Fairlie, P. Fletcher and C.K. Zachos, *J. Math. Phys.* **31**, 1088 (1990).
- [27] M. Daoud, Y. Hassouni and M. Kibler, in *Symmetries in Science X*, Eds. B. Gruber and M. Ramek (Plenum Press, New York, 1998); *Phys. Atom. Nuclei* **61**, 1821 (1998).
- [28] M.R. Kibler and M. Daoud, in: *Fundamental world of quantum chemistry*, Vol. III, Eds. E.J. Brändas and E.S. Kryachko (Kluwer, Dordrecht, 2004). M. Daoud and M. Kibler, *Phys. Lett.* **A321**, 147 (2004).